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The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices

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Abstract Given an ensemble of $N \times N$ random matrices, a natural question to ask is whether or not the empirical spectral measures of typical matrices converge to a limiting spectral measure as $N \to \infty$. While this has been proved for many thin patterned ensembles sitting inside all real symmetric matrices, frequently there is no nice closed form expression for the limiting measure. Further, current theorems provide few pictures of transitions between ensembles. We consider the ensemble of symmetric m-block circulant matrices with entries i.i.d.r.v. These matrices have toroidal diagonals periodic of period m. We view m as a "dial" we can "turn" from the thin ensemble of symmetric circulant matrices, whose limiting eigenvalue density is a Gaussian, to all real symmetric matrices, whose limiting eigenvalue density is a semi-circle. The limiting eigenvalue densities f_m show a visually stunning convergence to the semi-circle as $m \to \infty$, which we prove.

In contrast to most studies of patterned matrix ensembles, our paper gives explicit closed form expressions for the densities. We prove that f_m is the product of a Gaussian and a certain even polynomial of degree $2m - 2$; the formula is the same as that for the $m \times m$ Gaussian Unitary Ensemble (GUE). The proof is by derivation of the moments from the eigenvalue trace formula. The new feature, which allows us to obtain closed form expressions, is converting the central combinatorial problem in the moment calculation into an equivalent counting problem in algebraic topology. We end with a generalization of the m-block circulant pattern, dropping the assumption

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that the m random variables be distinct. We prove that the limiting spectral distribution exists and is determined by the pattern of the independent elements within an m-period, depending not only on the frequency at which each element appears, but also on the way the elements are arranged.

Keywords Limiting spectral measure · Circulant and Toeplitz matrices · Random matrix theory · Convergence · Method of moments · Orientable surfaces · Euler characteristic

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1 Introduction

1.1 History and Ensembles

Random matrix theory is the study of properties of matrices chosen according to some notion of randomness, which can range from taking the structurally independent entries as independent, identically distributed random variables to looking at subgroups of the classical compact groups under Haar measure. While the origins of the subject go back to Wishart's [44] investigations in statistics in the 1920s, it was Wigner's work [39–43] in the 1950s and Dyson's [11, 12] a few years later that showed its incredible power and utility, as random matrix ensembles successfully modeled the difficult problem of the distribution of energy levels of heavy nuclei. The next milestone was twenty years later, when Montgomery and Dyson [33] observed that the behavior of eigenvalues in certain random matrix ensembles correctly describe the statistical behavior of the zeros of the Riemann zeta function. The subject continues to grow, with new applications ranging from chemistry to network theory [31] to transportation systems [2, 26]. See [15, 21] for a history of the development of the subject and the discovery of some of these connections.

One of the most studied matrix ensembles is the ensemble of $N \times N$ real symmetric matrices. The N entries on the main diagonal and the $\frac{1}{2}N(N-1)$ entries in the upper right are taken to be independent, identically distributed random variables from a fixed probability distribution with density p having mean 0, variance 1, and finite higher moments. The remaining entries are filled in so that the matrix is real symmetric. Thus

$$\begin{aligned} \mathrm{Prob}(A) &= \prod_{1\le i\le j\le N} p(a_{ij}), \\ \mathrm{Prob}\bigl(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\bigr) &= \prod_{1\le i\le j\le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij})\, dx_{ij}. \end{aligned} \tag{1.1}$$

We want to understand the eigenvalues of A as we average over the family. Let $\delta(x-x_0)$ denote the shifted Delta functional (i.e., a unit point mass at x_0, satisfying $\int f(x)\delta(x-x_0)\,dx = f(x_0)$). To each A we associate its empirical spectral measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right). \tag{1.2}$$

Using the Central Limit Theorem, one readily sees that the correct scale to study the eigenvalues is on the order of $\sqrt{N}$.[1] The most natural question to ask is: How many normalized eigenvalues of a 'typical' matrix lie in a fixed interval as $N \to \infty$? Wigner proved that the answer is a semi-ellipse. This means that as $N \to \infty$, the empirical spectral measures of almost all A converge to the density of the semi-ellipse (with our normalization), whose density is

$$f_{\mathrm{Wig}}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-(\frac{x}{2})^2} & \text{if } |x| \le 2,\\ 0 & \text{otherwise.}\end{cases} \tag{1.3}$$

If we normalized the eigenvalues by $2\sqrt{N}$ instead of $\sqrt{N}$, we would obtain the density $\frac{2}{\pi}\sqrt{1-x^2}$ for $|x| \le 1$, which is known in the literature as Wigner's semi-circle law (although this density, too, is really a semi-ellipse).

As the eigenvalues of any real symmetric matrix are real, we can ask whether or not a limiting distribution exists for the density of normalized eigenvalues for other ensembles. There are many interesting families to study. McKay [32] proved that the limiting spectral measure for adjacency matrices attached to d-regular graphs on N vertices exists, and as $N \to \infty$, for almost all such graphs the associated measures converge to Kesten's measure

$$f_{\mathrm{Kesten},d}(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2}, & |x| \le 2\sqrt{d-1},\\ 0 & \text{otherwise}\end{cases} \tag{1.4}$$

(note that the measures may be scaled such that as $d \to \infty$ they converge to the semi-circle distribution).

This example and its behavior are typical for what we hope to find and prove. Specifically, we are looking for a thin subfamily that has different behavior but, as we fatten the ensemble to the full family of all real symmetric matrices, the limiting spectral measure converges to the semi-circle. Numerous researchers have studied a multitude of special, patterned matrices; we do not attempt to do this vast subject justice, but rather concentrate on a few ensembles closely related to our work.

All of the ensembles we consider here are linked ensembles (see [5]). A linked ensemble of $N \times N$ matrices is specified by a link function $L_N : \{1,2,\dots,N\}^2 \to S$

[1] $\sum_{i=1}^{N}\lambda_i^2 = \mathrm{Trace}(A^2) = \sum_{i,j\le N} a_{ij}^2$; as the mean is zero and the variance is one for each a_{ij}, this sum is of the order N^2, implying the average square of an eigenvalue is N.

to some set S. To $s \in S$, assign random variables x_s which are independent, identically distributed from a fixed probability distribution with density p having mean 0, variance 1, and finite higher moments. Set the (i, j)th entry of the matrix $a_{i,j} := x_{L_N(i,j)}$.[2] For some linked ensembles, including those we examine here, it is be more convenient to specify the ensemble not by the link function, but by the equivalence relation $\sim$ it induces on $\{1, 2, \dots, N\}^2$. A link function may be uncovered as the quotient map to the set of equivalence classes $\{1, 2, \dots, N\}^2 \twoheadrightarrow \{1, 2, \dots, N\}^2/\sim$. For example, the real symmetric ensemble is specified by the equivalence relation $(i, j) \sim (j, i)$.

One interesting thin linked ensemble is that of real symmetric Toeplitz matrices, which are constant along its diagonals. The limiting measure is close to but not a Gaussian (see [7, 10, 18]); however, in [29] the sub-ensemble where the first row is replaced with a palindrome is shown to have the Gaussian as its limiting measure. While the approach in [29] involves an analysis of an associated system of Diophantine equations, using Cauchy's interlacing property one can show that this problem is equivalent to determining the limiting spectral measure of symmetric circulant matrices (also studied in [9]).

While these and other ensembles related to circulant, Toeplitz, and patterned matrices are a very active area [3–5, 7–10, 18, 28, 29], of particular interest to us are ensembles of patterned matrices with a variable parameter controlling the symmetry. We desire to deform a family of matrices, starting off with a highly structured family and ending with the essentially structureless case of real symmetric matrices. This is in contrast to some other work, such as Kargin [24] (who studied banded Toeplitz matrices) and Jackson, Miller, and Pham [22] (who studied Toeplitz matrices whose first row had a fixed but arbitrarily number of palindromes). In these cases the ensembles are converging to the full Toeplitz ensemble (either as the band grows or the number of palindromes decreases). Other patterned matrices are possible. Meckes [30] considered matrices whose (i, j)th entry is $f(a_i a_j^{-1})$ for f a complex valued function on a finite Abelian group, and proved in many cases that the limiting behavior is either Gaussian or the sum of two Gaussians. Beckwith, Miller and Shen [6] study weighted patterned matrices, generalizing these investigations.

Our main ensemble is what we call the ensemble of *m-block circulant matrices*. A real symmetric circulant matrix (also called a symmetric circulant matrix) is a real symmetric matrix that is constant along diagonals and has first row $(x_0, x_1, x_2, \dots, x_2, x_1)$. Note that except for the main diagonal, a diagonal of length $N - k$ in the upper right is paired with a diagonal of length k in the bottom left, and all entries along these two diagonals are equal. We study block Toeplitz and circulant matrices with $m \times m$ blocks. The diagonals of such matrices are periodic of period m.

[2]For general linked ensembles, it may make more sense to weight the random variables by how often they occur in the matrix: $a_{i,j} := c_N |L_N^{-1}(\{L_N(i, j)\})|^{-1} x_{L_N(i,j)}$. For the real symmetric ensemble, this corresponds to weighting the entries along the diagonal by 2. In that case, and for the ensembles we examine here, this modification changes only lower order terms in the calculations of the limiting spectral measure.

Definition 1.1 (m-Block Toeplitz and Circulant Matrices) Let $m|N$. An $N \times N$ real symmetric m-block Toeplitz matrix is a Toeplitz matrix of the form

$$\begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{\frac{N}{m}-1} \\ B_{-1} & B_0 & B_1 & \cdots & B_{\frac{N}{m}-2} \\ B_{-2} & B_{-1} & B_0 & \cdots & B_{\frac{N}{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{1-\frac{N}{m}} & B_{2-\frac{N}{m}} & B_{3-\frac{N}{m}} & \cdots & B_0 \end{pmatrix},$$

with each B_i an $m \times m$ real matrix. An m-block circulant matrix is one of the above form for which $B_{-i} = B_{n-i}$.

We investigate real symmetric m-block Toeplitz and circulant matrices. In such matrices, a generic set of paired diagonals is composed of m independent entries, placed periodically; however, as the matrix is real symmetric, this condition occasionally forces additional entries on the paired diagonals of length $\frac{N}{2}$ to be equal.

For example, an 8×8 symmetric 2-block Toeplitz matrix has the form

$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_5 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_6 & d_5 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ c_7 & d_6 & c_5 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right), \tag{1.5}$$

while a 6×6 and an 8×8 symmetric 2-block circulant matrices have the form

$$\left(\begin{array}{cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & c_3 & d_2 & c_1 & d_0 \end{array}\right),$$
$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right); \tag{1.6}$$

Note for the 6×6 matrix that being real symmetric forces the paired diagonals of length $\frac{N}{2}$ (i.e., 3) to have just one and not two independent random variables. An equivalent viewpoint is that each ‘wrapped’ diagonal is periodic with period m and has m distinct random variables. Note that the diagonals are wrapped toroidally, and each such diagonal has N elements.

Clearly if $m = 1$ these ensembles reduce to the previous cases, and as $m \to \infty$ they approach the full family of real symmetric matrices; in other words, the circulant or Toeplitz structure vanishes as $m \to \infty$, but for any finite m there is additional structure. The goal of this paper is to determine the limiting spectral measures for these families and to quantify how the convergence to the semi-circle depends on m. We find an explicit closed form expression for the limiting spectral density of the m-block circulant family as a product of a Gaussian and a degree $2m - 2$ polynomial.

1.2 Results

Before stating our results, we must define the probability spaces where our ensemble lies and state the various types of convergence that we can prove. We provide full details for the m-block circulant matrices, as the related Toeplitz ensemble is similar. The following definitions and setup are standard, but are included for completeness. We paraphrase from [22, 29] with permission.

Fix m and for each integer N let $\Omega_{m,N}$ denote the set of m-block circulant matrices of dimension N. Define an equivalence relation $\simeq$ on $\{1, 2, \dots, N\}^2$. Say that $(i, j) \simeq (i', j')$ if and only if $a_{ij} = a_{i'j'}$ for all m-block circulant matrices; in other words, if

- $j - i \equiv j' - i' (\mathrm{mod}\ N)$ and $i \equiv i' (\mathrm{mod}\ m)$, or
- $j - i \equiv -(j' - i') (\mathrm{mod}\ N)$ and $i \equiv j' (\mathrm{mod}\ m)$.

Consider the quotient $\{1, 2, \dots, N\}^2 \twoheadrightarrow \{1, 2, \dots, N\}^2/\simeq$. This induces an injection $\mathbb{R}^{\{1,2,\dots,N\}^2/\simeq} \hookrightarrow \mathbb{R}^{N^2}$. The set $\mathbb{R}^{\{1,2,\dots,N\}^2/\simeq}$ has the structure of a probability space with the product measure of $p(x)\,dx$ with itself $|\{1, 2, \dots, N\}^2/\simeq|$ times, where dx is Lebesgue measure. We define the probability space $(\Omega_{m,N}, \mathcal{F}_{m,N}, \mathbb{P}_{m,N})$ to be its image in $\mathbb{R}^{N^2} = M_{N^2}(\mathbb{R})$ under the injection, with the same distribution.

To each $A_N \in \Omega_{m,N}$ we attach a measure by placing a point mass of size $\frac{1}{N}$ at each normalized eigenvalue $\lambda_i(A_N)$:

$$\mu_{m,A_N}(x)\,dx = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A_N)}{\sqrt{N}}\right)dx, \tag{1.7}$$

where $\delta(x)$ is the standard Dirac delta function; see Footnote 1 for an explanation of the normalization factor equaling $\sqrt{N}$. We call μ_{m,A_N} the normalized spectral measure associated with A_N.

Definition 1.2 (Normalized Empirical Spectral Distribution) Let $A_N \in \Omega_{m,N}$ have eigenvalues $\lambda_N \geq \cdots \geq \lambda_1$. The normalized empirical spectral distribution (the empirical distribution of normalized eigenvalues) $F_m^{\frac{A_N}{\sqrt{N}}}$ is defined by

$$F_m^{\frac{A_N}{\sqrt{N}}}(x) = \frac{\#\{i \leq N : \frac{\lambda_i}{\sqrt{N}} \leq x\}}{N}. \tag{1.8}$$

As $F_m^{A_N/\sqrt{N}}(x) = \int_{-\infty}^{x} \mu_{m,A_N}(t)\,dt$, we see that $F_m^{A_N/\sqrt{N}}$ is the cumulative distribution function associated with the measure μ_{n,A_N}. We are interested in the behavior of a typical $F_m^{A_N/\sqrt{N}}$ as we vary A_N in our ensembles $\Omega_{m,N}$ as $N \to \infty$.

Consider any probability space Ω_m which has the $\Omega_{m,N}$ as quotients. (The most obvious example is the independent product.) This paper build on a line of papers [18, 22, 29] concerning various Toeplitz ensembles which fix Ω_m to be the space of $\mathbb{N}$-indexed strings of real numbers picked independently from p, with quotient maps to each $\Omega_{m,N}$ mapping a string to a matrix whose free parameters come from an initial segment of the right length. There is no need for the specificities of this construction, so we consider the general case.

Definition 1.3 (Limiting Spectral Distribution) If as $N \to \infty$ we have $F_m^{A_N/\sqrt{N}}$ converging in some sense (for example, in probability or almost surely) to a distribution F_m, then we say F_m is the limiting spectral distribution of the ensemble.

We investigate the symmetric m-block Toeplitz and circulant ensembles. We may view these as structurally weakened real symmetric Toeplitz and circulant ensembles. When m is 1 we regain the Toeplitz (circulant) structure, while if $m = N$ we have the general real symmetric ensemble. If m is growing with the size of the matrix, we expect the eigenvalues to be distributed according to the semi-circle law, while for fixed m we expect to see new limiting spectral distributions.

Following the notation of the previous subsection, for each integer N, we let $\Omega_{m,N}^{(T)}$ and $\Omega_{m,N}^{(C)}$ denote the probability space of real symmetric m-block Toeplitz and circulant matrices of dimension N, respectively. We now state our main results.

Theorem 1.4 (Limiting Spectral Measures of Symmetric Block Toeplitz and Circulant Ensembles) *Let $m|N$.*

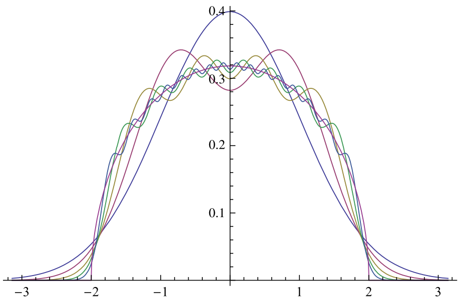
(1) *The characteristic function of the limiting spectral measure of the symmetric m-block circulant ensemble is*

$$\begin{aligned}\phi_m(t) &= \frac{1}{m} e^{-\frac{t^2}{2m}} e^{-\frac{t^2}{2m}} L_{m-1}^{(1)}\left(\frac{t^2}{m}\right) \\ &= e^{-\frac{t^2}{2m}} M\left(m+1, 2, -\frac{t^2}{m}\right),\end{aligned} \tag{1.9}$$

where $L_{m-1}^{(1)}$ is a generalized Laguerre polynomial and M a confluent hypergeometric function. The expression equals the spectral characteristic function for the $m \times m$ GUE. The limiting spectral density function (the Fourier transform of ϕ_m) is

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m-1} \frac{1}{(2r)!} \left(\sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s \right) (mx^2)^r. \tag{1.10}$$

Fig. 1 Plots for $f_1, f_2, f_4, f_8, f_{16}$ and the semi-circle density



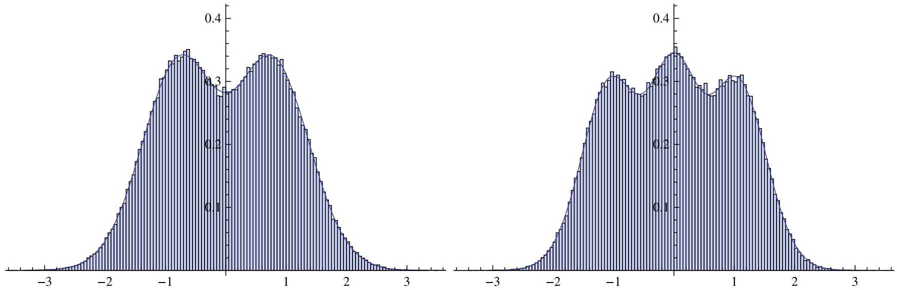


Fig. 2 (*Left*) Plot for f_2 and histogram of eigenvalues of 1000 symmetric period 2-block circulant matrices of size 400×400. (*Right*) Plot for f_3 and histogram of eigenvalues of 1000 symmetric period 3-block circulant matrices of size 402×402

For any fixed m, the limiting spectral density is the product of a Gaussian and an even polynomial of degree $2m-2$, and has unbounded support.

(2) *If m tends to infinity with N (at any rate) then the limiting spectral distribution of the symmetric m-block circulant and Toeplitz ensembles, normalized by rescaling x to $x/2$, converges to the semi-circle distribution; without the renormalization, the convergence is to a semi-ellipse, with density f_{Wig} (see (1.3)).*

(3) *As $m \to \infty$, the limiting spectral measures f_m of the m-block circulant ensemble converge uniformly and in L^p for any $p \geq 1$ to f_{Wig}, with $|f_m(x) - f_{\text{Wig}}(x)| \ll m^{-\frac{2}{9}+\epsilon}$ for any $\epsilon > 0$.*

(4) *The empirical spectral measures of the m-block circulant and Toeplitz ensembles converge weakly and in probability to their corresponding limiting spectral measures, and we have almost sure convergence if p is an even function.*

Figure 1 illustrates the convergence of the limiting measures to the semi-circle; numerical simulations (see Figs. 2, 3 and 4) illustrate the rapidity of the convergence. We see that even for small m, in which case there are only $\frac{mN}{2}$ non-zero entries in the adjacency matrices (though these can be any of the $N^2 - N$ non-diagonal entries of the matrix), the limiting spectral measure is close to the semi-circle. This behavior is similar to what happens with d-regular graphs, though in our case the convergence is faster and the support is unbounded for any finite m.

Finally, the limiting eigenvalue density for m-block circulant matrices is the same as the eigenvalue density of a certain Gaussian Hermitian ensemble. Specifically, we

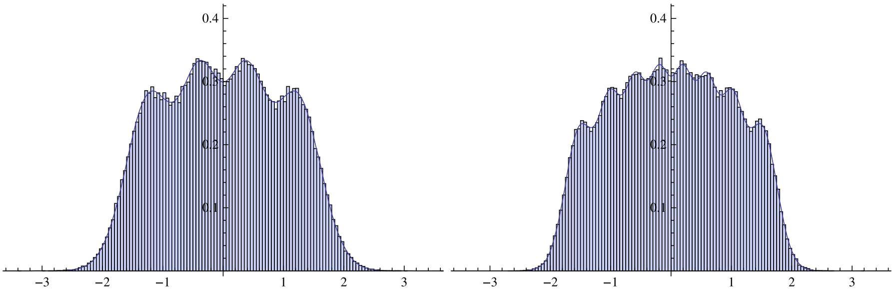


Fig. 3 (*Left*) Plot for f_4 and histogram of eigenvalues of 1000 symmetric period 4-block circulant matrices of size 400×400. (*Right*) Plot for f_8 and histogram of eigenvalues of 1000 symmetric period 8-block circulant matrices of size 400×400

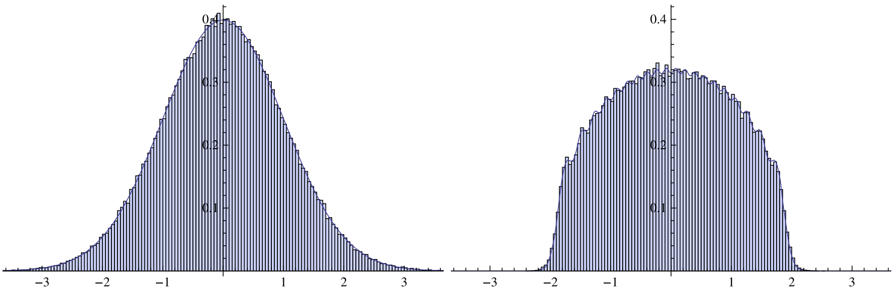


Fig. 4 (*Left*) Plot for f_1 and histogram of eigenvalues of 1000 symmetric period 1-block circulant matrices of size 400×400. (*Right*) Plot for f_{20} and histogram of eigenvalues of 1000 symmetric period 20-block circulant matrices of size 400×400

consider $m \times m$ Hermitian matrices with off-diagonal entries picked independently from a complex Gaussian with density function $p(z) = \frac{1}{\pi} e^{-|z|^2}$, and diagonal entries picked independently from a real Gaussian of mean 0 and variance 1. We provide a heuristic for why these densities are the same in Sect. 5.1; see also [46] (especially Sect. 5.2) for a proof.

Our results generalize to related ensembles. For example, the (wrapped) diagonals of our m-block circulant ensembles have the following structure (remember we assume $m|N$):

$$(b_{1,j}, b_{2,j}, \dots, b_{m,j}, b_{1,j}, b_{2,j}, \dots, b_{m,j}, \dots, b_{1,j}, b_{2,j}, \dots, b_{m,j}). \tag{1.11}$$

Note that we have a periodic repeating block of size m with m independent random variables; for brevity, we denote this structure by

$$(d_1, d_2, \dots, d_m). \tag{1.12}$$

Similar arguments handle other related ensembles, such as the subfamily of period m-circulant matrices in which some entries within the period are forced to be equal. Interesting comparisons are $(d_1, d_2) = (d_1, d_2, d_1, d_2)$ versus (d_1, d_1, d_2, d_2)

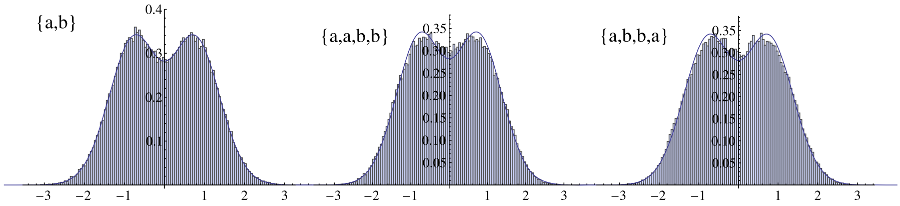


Fig. 5 Eigenvalue histograms for 200 patterned m-block circulant, 1200×1200 matrices; the first is the pattern $\{a, b\}$ (which may also be written as $\{a, b, a, b\}$ or $\{d_1, d_2\}$), the second is $\{a, a, b, b\}$ and the third is $\{a, b, b, a\}$

or (d_1, d_2, d_2, d_1). While it is a natural guess that the limiting spectral measure is determined solely by the frequency at which each letter appears, this is false as Theorem 1.5 shows, though the differences are often so small that visually different patterns seem to give rise to the same limiting distribution (see Fig. 5 and Tables 1, 2 and 3 in Appendix B).

Theorem 1.5 *Let $\mathcal{P} = (d_{i_1}, d_{i_2}, \dots, d_{i_m})$ where each $d_{i_j} \in \{d_1, \dots, d_\nu\}$ and each d_i occur exactly r_i times in the pattern $\mathcal{P}$, with $r_1 + \cdots + r_\nu = m$; equivalently, $\mathcal{P}$ is a permutation of $(d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_\nu, \dots, d_\nu)$ with r_i copies of d_i. Modify the $N \times N$ period m-block circulant matrices by replacing the pattern $(d_1, d_2, \dots, d_m)$ with $\mathcal{P}$ (remember $m|N$). Then for any $\mathcal{P}$ as $N \to \infty$ the limiting spectral measure exists. The resulting measure does not depend solely on the frequencies of the letters in the pattern but also on their locations; in particular, while the fourth moments of the measures associated with $\{d_1, d_2, d_1, d_2\}$ and $\{d_1, d_1, d_2, d_2\}$ are equal (interestingly, the fourth moment of any pattern depends only on the frequencies), the sixth moments differ.*

We prove our main results using the method of moments. As the proof of Theorem 1.5 is similar to that of Theorem 1.4, we just sketch the ideas and computations in Appendix B. For our ensembles, we first show that the average of the kth moments over our ensemble converge to the moments of a probability density. By studying the variance or fourth moment of the difference of the moments of the empirical spectral measures and the limits of the average moments, we obtain the various types of convergence by applications of Chebyshev's inequality and the Borel–Cantelli Lemma. These arguments are similar to previous works in the literature, and yield only the existence of the limiting spectral measure.

Unlike other works for related ensembles, however, we are able to obtain explicit closed form expressions for the moments for the symmetric m-block circulant ensemble. This should be compared to the Toeplitz ensemble case, where previous studies could only relate these moments to volumes of Eulerian solids or solutions to systems of Diophantine equations. Similarly to other ensembles, we show that the only contribution in the limit is when $k = 2\ell$ and the indices are matched in pairs with opposite orientation. We may view this as a 2ℓ-gon with vertices $(i_1, i_2), (i_2, i_3), \dots, (i_{2\ell}, i_1)$. The first step is to note that when $m = 1$, similarly to the circulant and palindromic Toeplitz ensembles, each matching contributes 1; as there are $(2\ell - 1)!!$ ways to match

2ℓ objects in pairs, and as $(2\ell - 1)!!$ is the 2ℓth moment of the standard normal, this yields the Gaussian behavior. For general m, the key idea is to look at the dual picture. Instead of matching indices we match edges. In the limit as $N \to \infty$, the only contribution occurs when the edges are matched in pairs with opposite orientation. Topologically, these are exactly the pairings which give orientable surfaces. If g is the genus of the associated surface, then the matching contributes m^{-2g}. Harer and Zagier [19] determined formulas for $\varepsilon_g(\ell)$, the number of matchings that form these orientable surfaces. This yields the $N \to \infty$ limit of the average 2ℓth moment is

$$\sum_{g=0}^{\lfloor \frac{\ell}{2} \rfloor} \varepsilon_g(\ell) m^{-2g}. \tag{1.13}$$

After some algebra, we express the characteristic function (which is the inverse Fourier transform; see Footnote 3) of the limiting spectral measure as a certain term in the convolution of the associated generating function of the ε_g's and the normal distribution, which we can compute using Cauchy's residue theorem. Taking the Fourier transform (appropriately normalized) yields an explicit, closed form expression for the density. We note that the same formulas arise in investigations of the moments for Gaussian ensembles; see Sect. 1.6 of [16] and [46] (as well as the references therein) for additional comments and examples.

The paper is organized as follows. In Sect. 2 we describe the method of proof and derive useful expansions for the moments in terms of quantities from algebraic topology. We use these in Sect. 3 to determine the limiting spectral measures, and show convergence in Sect. 4. We conclude in Sect. 5 with a description of future work and related results. Appendix A provides some needed estimates for proving the rate of convergence in Theorem 1.4, and we conclude in Appendix B with a discussion of the proof of Theorem 1.5 (see [45] for complete details).

2 Moments Preliminaries

In this section we investigate the moments of the associated spectral measures. We first describe the general framework of the convergence proofs and then derive useful expansions for the average moments for our ensemble for each N (Lemma 2.2). The average odd moments are easily seen to vanish, and we find a useful expansion for the $2k$th moment in Lemma 2.4, relating this moment to the number of pairings of the edges of a $2k$-gon giving rise to a genus g surface.

2.1 Markov's Method of Moments

For the eigenvalue density of a particular $N \times N$ symmetric m-block circulant matrix A, we use the redundant notation $\mu_{m,A,N}(x)\,dx$ (to emphasize the N dependence), setting

$$\mu_{A,N}(x)\,dx := \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right) dx. \tag{2.1}$$

To prove Theorem 1.4, we must show

(1) as $N \to \infty$ a typical matrix has its spectral measure close to the system average;
(2) these system averages converge to the claimed measures.

The second claim follows easily from Markov's Method of Moments, which we now briefly describe. To each integer $k \geq 0$ we define the random variable $X_{k;m,N}$ on Ω_m by

$$X_{k;m,N}(A) = \int_{-\infty}^{\infty} x^k \, dF_m^{\frac{A_N}{\sqrt{N}}}(x); \tag{2.2}$$

note that this is the kth moment of the measure $\mu_{m,A,N}$.

Our main tool to understand the average over all A in our ensemble of the $F_m^{A_N/\sqrt{N}}$'s is the Moment Convergence Theorem (see [36] for example); while the analysis in [29] was simplified by the fact that the convergence was to the standard normal, similar arguments (see also [22]) hold in our case as the growth rate of the moments of our limiting distribution implies that the moments uniquely determine a probability distribution.

Theorem 2.1 (Moment Convergence Theorem) *Let $\{F_N(x)\}$ be a sequence of distribution functions such that the moments*

$$M_{k;N} = \int_{-\infty}^{\infty} x^k \, dF_N(x) \tag{2.3}$$

exist for all k. Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of moments that uniquely determine a probability distribution, and denote the cumulative distribution function by Ψ. If $\lim_{N\to\infty} M_{k,N} = M_k$ *then* $\lim_{N\to\infty} F_N(x) = \Psi(x)$.

We will see that the average moments uniquely determine a measure, and will be left with proving that a typical matrix has a spectral measure close to the system average. The nth moment of A's measure, given by integrating x^n against $\mu_{m,A,N}$, is

$$M_{n;m}(A,N) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i(A)}{\sqrt{N}}\right)^n = \frac{1}{N^{\frac{n}{2}+1}}\sum_{i=1}^{N}\lambda_i^n(A). \tag{2.4}$$

We define

$$M_{n;m}(N) := \mathbb{E}\big(M_{n;m}(A,N)\big), \tag{2.5}$$

and set

$$M_{n;m} := \lim_{N\to\infty} M_{n;m}(N) \tag{2.6}$$

(we will show later that the limit exists). By $\mathbb{E}(M_{n;m}(A,N))$, we mean the expected value of $M_{n;m}(A,N)$ for a random symmetric m-block circulant matrix $A \in \Omega_{m,N}$.

2.2 Moment Expansion

We use a standard method to compute the moments. By the eigenvalue trace lemma,

$$\operatorname{Tr}\bigl(A^n\bigr) = \sum_{i=1}^{N} \lambda_i^n, \tag{2.7}$$

so

$$M_{n;m}(A, N) = \frac{1}{N^{\frac{n}{2}+1}} \operatorname{Tr}\bigl(A^n\bigr). \tag{2.8}$$

Expanding out $\operatorname{Tr}(A^n)$,

$$M_{n;m}(A, N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}, \tag{2.9}$$

so by linearity of expectation,

$$M_{n;m}(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}). \tag{2.10}$$

Recall that we have defined the equivalence relation $\simeq$ on $\{1, 2, \dots, N\}^2$ by $(i, j) \simeq (i', j')$ if and only if $a_{ij} = a_{i'j'}$ for all real symmetric m-block circulant matrices. That is, $(i, j) \simeq (i', j')$ if and only if

- $j - i \equiv j' - i' (\operatorname{mod} N)$ and $i \equiv i' (\operatorname{mod} m)$, or
- $j - i \equiv -(j' - i') (\operatorname{mod} N)$ and $i \equiv j' (\operatorname{mod} m)$.

For each term in the sum in (2.10), $\simeq$ induces an equivalence relation $\sim$ on $\{(1, 2), (2, 3), \dots, (n, 1)\}$ by its action on $\{(i_1, i_2), (i_2, i_3), \dots, (i_n, i_1)\}$. Let $\eta(\sim)$ denote the number of n-tuples with $0 \le i_1, \dots, i_n \le N$ whose indices inherit $\sim$ from $\simeq$. Say $\sim$ splits up $\{(1, 2), (2, 3), \dots, (n, 1)\}$ into equivalence classes with sizes $d_1(\sim), \dots, d_l(\sim)$. Because the entries of our random matrices are independent, identically distributed,

$$\mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) = m_{d_1(\sim)} \cdots m_{d_l(\sim)}, \tag{2.11}$$

where the m_d are the moments of p. Thus, we may write

$$M_{n;m}(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \tag{2.12}$$

As p has mean 0, $m_{d_1(\sim)} \cdots m_{d_l(\sim)} = 0$ unless all of the d_j's are greater than 1. So all the terms in the above sum vanish except for those coming from a relation $\sim$ which matches at least in pairs.

The $\eta(\sim)$ denotes the number of solutions modulo N following the system of Diophantine equations: whenever $(s, s + 1) \sim (t, t + 1)$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t (\operatorname{mod} N)$ and $i_s \equiv i_t (\operatorname{mod} m)$, or

- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t)(\bmod N)$ and $i_s \equiv i_{t+1}(\bmod m)$.

This system has at most $2^{n-l}N^{l+1}$ solutions, a bound we obtain by completely ignoring the (mod m) constraints (see also [29]). Specifically, we pick one difference $i_{s+1} - i_s$ from each congruence class of $\sim$ freely, and we are left with at most 2 choices for the remaining ones. Finally, we pick i_1 freely, and this now determines all the $i_s = i_1 + \sum_{s'<s}(i_{s'+1} - i_{s'})$. This method will not always produce a legitimate solution, even without the (mod m) constraints, but it suffices to give an upper bound on the number of solutions.

When n is odd, say $n = 2k + 1$, then l is at most k. Thus $\frac{1}{N^{\frac{n}{2}+1}}\eta(\sim) \le \frac{1}{N^{k+\frac{3}{2}}}2^{n-l}N^{l+1} \le \frac{1}{N^{k+\frac{3}{2}}}2^{n-l}N^{k+1} = \frac{1}{\sqrt{N}}2^{n-l} = O_n(\frac{1}{\sqrt{N}})$. This implies the odd moments vanish in the limit, as

$$M_{2k+1;m}(N) = O_k\left(\frac{1}{\sqrt{N}}\right). \tag{2.13}$$

When n is even, say $n = 2k$, then l is at most k. If $l < k$, then $l \le k - 1$, and we have, similarly to the above, $\frac{1}{N^{\frac{n}{2}+1}}\eta(\sim) \le \frac{1}{N^{k+1}}2^{n-l}N^{l+1} \le \frac{1}{N^{k+1}}2^{n-l}N^k = \frac{1}{N}2^{n-l} = O_n(\frac{1}{N})$. If $l = k$, then the entries are exactly matched in pairs, that is, all the $d_j = 2$. As p has variance 1 (i.e., $m_2 = 1$), the formula for the even moments, (2.12), becomes

$$M_{2k;m}(N) = \frac{1}{N^{k+1}}\sum_{\sigma}\eta(\sigma) + O_k\left(\frac{1}{N}\right). \tag{2.14}$$

We have changed notation slightly. The sum is now over pairings σ on $\{(1,2),(2,3),\dots,(n,1)\}$, which we may consider as functions (specifically, involutions with no fixed points) as well as equivalence relations. We have thus shown the following:

Lemma 2.2 *For the ensemble of symmetric m-block circulant matrices,*

$$\begin{aligned} M_{2k+1;m}(N) &= O_k\left(\frac{1}{\sqrt{N}}\right), \\ M_{2k;m}(N) &= \frac{1}{N^{k+1}}\sum_{\sigma}\eta(\sigma) + O_k\left(\frac{1}{N}\right), \end{aligned} \tag{2.15}$$

where the sum is over pairings σ on $\{(1,2),(2,3),\dots,(n,1)\}$. In particular, as $N \to \infty$, the average odd moment is zero.

2.3 Even Moments

We showed that the odd moments go to zero like $\frac{1}{\sqrt{N}}$ as $N \to \infty$; we now calculate the $2k$th moments. From Lemma 2.2, the only terms which contribute in the limit are those in which the $a_{i_s i_{s+1}}$'s are matched in pairs. We can think of the pairing as a pairing of the edges of a $2k$-gon with vertices $1, 2, \dots, 2k$ and edges $(1,2),(2,3),\dots,(2k,1)$. The vertices are labeled $i_1,\dots,i_{2k}$ and the edges are labeled $a_{i_1 i_2},\dots,a_{i_{2k} i_1}$. See Fig. 6.

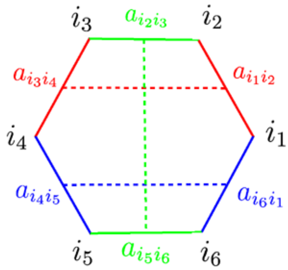


Fig. 6 Diagram for a pairing arising in computing the 6th moment

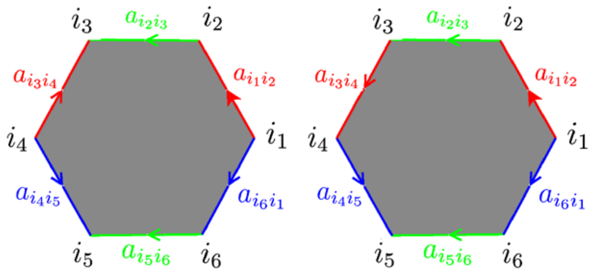


Fig. 7 Some possible orientations of paired edges for the 6-gon

Note that this is dual to the diagrams for pairings that appear in [18, 29], in which the $a_{i_s i_{s+1}}$'s are represented as vertices. For more on such an identification and its application in determining moments for random matrix ensembles, see [16] (Sect. 1.6) and [46].

If $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are paired, we have either

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod N$ and $i_s \equiv i_t \pmod m$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod N$ and $i_s \equiv i_{t+1} \pmod m$.

We think of these two cases as pairing $(s, s+1)$ and $(t, t+1)$ with the same or opposite orientation, respectively. For example, in Fig. 7 the hexagon on the left has all edges paired in opposite orientation, and the one on the right has all but the red edges paired in opposite orientation.

We now dramatically reduce the number of pairings we must consider by showing that the only pairings which contribute in the limit are those in which all edges are paired with opposite orientation. Topologically, these are exactly the pairings which give orientable surfaces [19, 20]. This result and its proof are a minor modification of their analogs in the Toeplitz and palindromic Toeplitz cases [18, 22, 29].

Lemma 2.3 *Consider a pairing σ with orientations ε_j. If any ε_j is equal to* 1, *then the pairing contributes $O_k(\frac{1}{N})$.*

Proof The size of the contribution is equal to the number of solutions to the k equations

$$i_{s+1} - i_s \equiv \varepsilon_j (i_{\sigma(s)+1} - i_{\sigma(s)}) \pmod N, \tag{2.16}$$

as well as some $\pmod m$ equations, divided by N^{k+1}. We temporarily ignore the $\pmod m$ constraints and bound the contribution from above by the number of solutions to the (mod N) equations over N^{k+1}. Because the i_s's are restricted to the values $1, 2, \ldots, N$, we can consider them as elements of $\mathbb{Z}/N\mathbb{Z}$, and we now notate the (mod N) congruences with equality.

The pairing puts the numbers $1, 2, \ldots, 2k$ into k equivalence classes of size two; arbitrarily order the equivalence classes and pick an element from each to call s_j, naming the other element $t_j = \sigma(s_j)$.

Our $\mathbb{Z}/N\mathbb{Z}$ equations now look like

$$i_{s_j+1} - i_{s_j} = \varepsilon_j (i_{t_j+1} - i_{t_j}) \bmod N. \tag{2.17}$$

Defining

$$\begin{aligned} x_j &:= i_{s_j+1} - i_{s_j} \\ y_j &:= i_{t_j+1} - i_{t_j}, \end{aligned} \tag{2.18}$$

our equations now look like $x_j = \varepsilon_j y_j$. Thus

$$0 = \sum_{s=1}^{2k} i_{s+1} - i_s = \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j = \sum_{j=1}^{k} (\varepsilon_j + 1) y_j.$$

If any one of the $\varepsilon_j = 1$, this gives a non-trivial relation among the y_j, and we lose a degree of freedom. We may choose $k - 1$ of the y_j freely (in $\mathbb{Z}/N\mathbb{Z}$), and we are left with 1 or possibly 2 choices for the remaining y_j (depending on the parity of N). The x_j's are now determined as well, so $i_{s+1} - i_s$ is now determined for every s. If we choose i_1 freely, this now determines all the $i_s = i_1 + \sum_{s'<s} (i_{s'+1} - i_{s'})$. Thus, we have at most $N^{k-1} \cdot 2 \cdot N = 2N^k$ solutions to (2.16). So the contribution from a pairing with a positive sign is at most $O_k(2\frac{N^k}{N^{k+1}}) = O_k(\frac{1}{N})$. (The reason for the big-Oh constant depending on k is that if some of the different pairs have the same value, we might not have k copies of the second moment but instead maybe four second moments and two eighth moments; however, the contribution is trivially bounded by $\max_{1 \le \ell \le k} (1 + m_{2\ell})^k$, where $m_{2\ell}$ is the 2ℓth moment of p.) □

Thus we have

$$M_{2k;m}(N) = \sum_{\sigma} w(\sigma) N^{-(k+1)} + O_k\left(\frac{1}{N}\right), \tag{2.19}$$

where $w(\sigma)$ denotes the number of solutions to

$$i_{j+1} - i_j \equiv -(i_{\sigma j+1} - i_{\sigma j}) \bmod N \tag{2.20}$$

and

$$i_j \equiv i_{\sigma(j)+1}, \qquad i_{j+1} \equiv i_{\sigma(j)} \bmod m \tag{2.21}$$

(the second (mod m) constraint is redundant). We discuss how to evaluate this moment in closed form, culminating in Lemma 2.4.

We now consider a given pairing as a topological identification (see [20] for an exposition of the standard theory); this is the crux of our argument. Specifically, consider a $2k$-gon with the interior filled in (homeomorphic to the disk), and identify the paired edges with opposite orientation. Under the identification, some vertices are identified; let v denote the number of vertices in the quotient.

Consider the $(\mathbb{Z}/N\mathbb{Z})$-submodule $\mathcal{A}$ of $(\mathbb{Z}/N\mathbb{Z})^{2k}$ in which the (mod N) constraints hold. We have that $\mathcal{A}$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{k+1}$. Specifically, we may freely choose the value of exactly half of the differences $i_{s+1} - i_s$, and then the rest are determined. Because all the pairings are with opposite orientation, these "differences" sum to zero, so they are actually realizable as differences. Now choose i_1 freely, and the rest of the $i_s = i_1 + \sum_{s'<s}(i_{s'+1} - i_{s'})$ are determined.

Let $\bar{\mathcal{A}}$ denote the quotient of $\mathcal{A}$ in which everything is reduced modulo m, and consider the $(\mathbb{Z}/m\mathbb{Z})$-submodule $B \subseteq \bar{\mathcal{A}}$ in which the modulo m constraints hold. By (2.21), we can see that the labels at two vertices of our $2k$-gon are forced to be congruent (mod m) if and only if the vertices are identified in the quotient, and these are all the (mod m) constraints. In other words, $\mathcal{B}$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^v$. An element of $\mathcal{A}$ for which the (mod m) constraints also hold is exactly one in the preimage of $\mathcal{B}$. We have m^v choices for an element in $\mathcal{B}$, and there are $(\frac{N}{m})^{k+1}$ ways to lift such an element to an element of A in its fiber. Thus, the equations have a total of $m^v(\frac{N}{m})^{k+1} = m^{-(k+1-v)}N^{k+1}$, so the pairing has a contribution of $m^{-(k+1-v)}$.

Let X be the 2-dimensional cell complex described by the pairing σ of the edges of the $2k$-gon. Because all edges were paired in the reverse direction, X is an orientable surface. After identifications, the complex we have described has 1 face, k edges, and, say, v vertices. If we denote by g the genus of the surface, we obtain two expressions for the Euler characteristic of X. By the standard (homological) definition of Euler characteristic, we have $\chi(X) = 1 - k + v$. On the other hand, for a genus g surface X, $\chi(X) = 2 - 2g$ [20]. Equating and rearranging,

$$2g = k + 1 - v. \tag{2.22}$$

Thus the pairing σ contributes m^{-2g}, and we have shown the following:

Lemma 2.4 *For the ensemble of symmetric m-block circulant matrices*,

$$M_{2k;m}(N) = \sum_g \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right), \tag{2.23}$$

where $\varepsilon_g(k)$ denotes the number of pairings of the edges of a $2k$-gon which give rise to a genus g surface.

3 Determining the Limiting Spectral Measures

We prove parts (1) and (2) of Theorem 1.4. Specifically, we derive the density formula for the limiting spectral density of symmetric m-block circulant matrices. We show that, if m grows at any rate with N, then the limiting spectral density is the semi-circle for both the symmetric m-block circulant and Toeplitz ensembles.

3.1 The Limiting Spectral Measure of the Symmetric m-Block Circulant Ensemble

Proof of Theorem 1.4(1) By deriving an explicit formula, we show that the limiting spectral density function f_m of the real symmetric m-block circulant ensemble is equal to the spectral density function of the $m \times m$ GUE.

From Lemma 2.4, the $N \to \infty$ limit of the average $2k$th moment equals

$$M_{2k;m} = \sum_{g=0}^{\lfloor \frac{k}{2} \rfloor} \varepsilon_g(k) m^{-2g}, \tag{3.1}$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $2k$-gon giving rise to a genus g surface. Harer and Zagier [19] give formulas for the $\varepsilon_g(k)$. They prove

$$\varepsilon_g(k) = \frac{(2k)!}{(k+1)!(k-2g)!} \times \left(\text{coefficient of } x^{2g} \text{ in } \left(\frac{\frac{x}{2}}{\tanh(\frac{x}{2})} \right)^{k+1} \right) \tag{3.2}$$

and

$$\sum_{g=0}^{\lfloor \frac{k}{2} \rfloor} \varepsilon_g(k) r^{k+1-2g} = (2k-1)!!c(k,r), \tag{3.3}$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} = \left(\frac{1+x}{1-x} \right)^r. \tag{3.4}$$

Thus, we may write

$$M_{2k;m} = m^{-(k+1)}(2k-1)!!c(k,m). \tag{3.5}$$

We construct the characteristic function[3] of the limiting spectral distribution. Let X_m be a random variable with density f_m. Then (remembering that the odd moments vanish)

$$\begin{aligned} \phi_m(t) = \mathbb{E}\big[e^{itX_m}\big] &= \sum_{\ell=0}^{\infty} \frac{(it)^\ell M_{\ell;m}}{\ell!} \\ &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k;m}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} m^{-(k+1)}(2k-1)!!c(k,m)\big(-t^2\big)^k. \end{aligned} \tag{3.6}$$

[3]The characteristic function is $\phi_m(t) = \mathbb{E}[e^{itX_m}] = \int_{-\infty}^{\infty} f_m(x) e^{itx} dx$. This is the inverse Fourier transform of f_m.

In order to obtain a closed form expression, we rewrite the characteristic function as

$$\phi_m(t) = \frac{1}{m}\sum_{k=0}^{\infty} c(k,m)\frac{1}{k!}\left(\frac{-t^2}{2m}\right)^k, \tag{3.7}$$

using $(2k-1)!! = \frac{(2k)!}{2^k k!}$. The reason for this is that we can interpret the above as a certain coefficient in the convolution of two known generating functions, which can be isolated by a contour integral. Specifically, consider the two functions

$$\begin{aligned} F(y) &:= \frac{1}{2y}\left(\left(\frac{1+y}{1-y}\right)^m - 1\right) = \sum_{k=0}^{\infty} c(k,m) y^k \quad \text{and} \\ G(y) &:= e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}. \end{aligned} \tag{3.8}$$

Note that $\phi_m(t)$ is the function whose power series is the sum of the products of the kth coefficients of $G(-\frac{y^2}{2m})$ (which is related to the exponential distribution) and $F(y)$ (which is related to the generating function of the $\varepsilon_g(k)$). Thus, we may use a multiplicative convolution to find a formula for the sum. By Cauchy's residue theorem, integrating $F(z^{-1})G(-\frac{t^2 z}{2m})z^{-1}$ over the circle of radius 2 yields

$$\phi_m(t) = \frac{1}{2\pi i m}\oint_{|z|=2} F\left(z^{-1}\right) G\left(-\frac{t^2 z}{2m}\right)\frac{dz}{z}, \tag{3.9}$$

since the constant term in the expansion of $F(z^{-1})G(-\frac{t^2 z}{2m})$ is exactly the sum of the products of coefficients for which the powers of y in $F(y)$ and $G(y)$ are the same.[4] We are integrating along the circle of radius 2 instead of the unit circle to have the pole inside the circle and not on it. Thus

$$\begin{aligned} \phi_m(t) &= \frac{1}{2\pi i m}\oint_{|z|=2} \frac{1}{2z^{-1}}\left(\left(\frac{1+z^{-1}}{1-z^{-1}}\right)^m - 1\right) e^{-\frac{t^2 z}{2m}}\frac{dz}{z} \\ &= \frac{1}{4\pi i m}\oint_{|z|=2}\left(\left(\frac{z+1}{z-1}\right)^m - 1\right) e^{-\frac{t^2 z}{2m}}\, dz \\ &= \frac{e^{-\frac{t^2}{2m}}}{4\pi i m}\oint_{|z|=2}\left(\left(1+\frac{2}{z-1}\right)^m - 1\right) e^{-\frac{t^2(z-1)}{2m}}\, dz \\ &= \frac{e^{-\frac{t^2}{2m}}}{4\pi i m}\oint_{|z|=2}\sum_{l=0}^{m}\binom{m}{l}\left(\frac{2}{z-1}\right)^l \sum_{s=0}^{\infty}\frac{1}{s!}\left(\frac{-t^2}{2m}\right)^s (z-1)^s\, dz \end{aligned}$$

[4] All functions are meromorphic in the region with finitely many poles; thus, the contour integral yields the sum of the residues. See e.g. [35].

$$- \frac{e^{-\frac{t^2}{2m}}}{4\pi i m} \oint_{|z|=2} e^{-\frac{t^2(z-1)}{2m}} \, dz. \tag{3.10}$$

By Cauchy's Residue Theorem the second integral vanishes and the only surviving terms in the first integral are when $l - s = 1$, whose coefficient is the residue. Thus

$$\begin{aligned} \phi_m(t) &= \frac{e^{-\frac{t^2}{2m}}}{2m} \sum_{l=1}^{m} \binom{m}{l} 2^l \frac{1}{(l-1)!} \left(\frac{-t^2}{2m} \right)^{l-1} \\ &= \frac{1}{m} e^{-\frac{t^2}{2m}} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m} \right)^{l-1} = \frac{1}{m} e^{-\frac{t^2}{2m}} L_{m-1}^{(1)} \left(\frac{t^2}{m} \right), \end{aligned} \tag{3.11}$$

which equals the spectral density function of the $m \times m$ GUE (see [27]).

As the density and the characteristic function are a Fourier transform pair, each can be recovered from the other through either the Fourier or the inverse Fourier transform (see for example [34, 35]). Since the characteristic function is given by

$$\phi_m(t) = \mathbb{E}\big[e^{itX_m}\big] = \int_{-\infty}^{\infty} e^{itx} f_m(x) \, dx \tag{3.12}$$

(where X_m is a random variable with density f_m), the density is regained by the relation

$$f_m(x) = \widehat{\phi_m}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_m(t) \, dt. \tag{3.13}$$

Taking the Fourier transform of the characteristic function $\phi_m(t)$, and interchanging the sum and the integral, we get

$$\begin{aligned} f_m(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{t^2}{2m}}}{m} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m} \right)^{l-1} e^{-itx} \, dt \\ &= -\frac{1}{2\pi} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} (-m)^{-l} \int_{-\infty}^{\infty} t^{2(l-1)} e^{-\frac{t^2}{2m}} e^{-itx} \, dt \\ &= -\frac{1}{2\pi} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} (-m)^{-l} I_m. \end{aligned} \tag{3.14}$$

Completing the square in the integrand of I_m, we obtain

$$I_m = e^{-\frac{mx^2}{2}} \int_{-\infty}^{\infty} t^{2(l-1)} \exp\left(-\frac{1}{2} \left(\frac{t}{\sqrt{m}} + i\sqrt{m}x \right)^2 \right) dt. \tag{3.15}$$

Changing variables by $y = \frac{1}{\sqrt{m}} t + i\sqrt{m}x$, $dy = \frac{1}{\sqrt{m}} \, dt$, we find I_m equals

$$
\begin{aligned}
I_m &= e^{-\frac{mx^2}{2}} \int_{-\infty}^{\infty} (y - i\sqrt{m}x)^{2(l-1)} (\sqrt{m})^{2(l-1)} e^{-\frac{y^2}{2}} \sqrt{m}\, dy \\
&= e^{-\frac{mx^2}{2}} m^{l-\frac{1}{2}} \sum_{s=0}^{2(l-1)} \binom{2(l-1)}{s} (-i\sqrt{m}x)^{2(l-1)-s} \int_{-\infty}^{\infty} y^s e^{-\frac{y^2}{2}}\, dy. \quad (3.16)
\end{aligned}
$$

The integral above is the sth moment of the Gaussian, and is $\sqrt{2\pi}(s-1)!!$ for even s and 0 for odd s. Since the odd s terms vanish, we replace the variable s with $2s$ and sum over $0 \le s \le (l-1)$. We find

$$
I_m = \sqrt{2\pi} e^{-\frac{mx^2}{2}} m^{l-\frac{1}{2}} \sum_{s=0}^{l-1} \binom{2(l-1)}{2s} \left(-mx^2\right)^{l-1-s} (2s-1)!!. \quad (3.17)
$$

Substituting this expression for I_m into (3.14) and making the change of variables $r = l - 1 - s$, we find that the density is

$$
f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m-1} \frac{1}{(2r)!} \left(\sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s \right) \left(mx^2\right)^r. \quad (3.18)
$$

This completes the proof of Theorem 1.4(1). □

3.2 The $m \to \infty$ Limit and the Semi-circle

Before proving Theorem 1.4(2), we first derive expressions for the limits of the average moments of the symmetric m-block Toeplitz ensemble. We sketch the argument. Though the analysis is similar to its circulant cousin, it presents more difficult combinatorics. Because diagonals do not "wrap around", certain diagonals are better to be on than others. Consequently, the Diophantine obstructions of [18] are present. The problems are the matchings with "crossings", or, in topological language, those matchings which give rise to tori with genus $g \ge 1$ as opposed to spheres with $g = 0$. For a detailed analysis of the Diophantine obstructions and how the added circulant structure fixes them, see [18] and [29]. Fortunately, it is easy to show that the contributions to the $2k$th moment of the symmetric m-block Toeplitz distribution from the non-crossing (i.e, the spherical matchings or, in the language of [5], the Catalan words) are unhindered by Diophantine obstructions and thus contribute fully. The number of these matchings is C_k, which is the kth Catalan number $\frac{1}{k+1}\binom{2k}{k}$ as well as the $2k$th moment of the Wigner density

$$
f_{\mathrm{Wig}}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - (\frac{x}{2})^2} & \text{if } |x| \le 2, \\ 0 & \text{otherwise.} \end{cases} \quad (3.19)
$$

Note that with this, normalization we have a semi-ellipse and not a semi-circle; to obtain the semi-circle, we normalize the eigenvalues by $2\sqrt{N}$ and not $\sqrt{N}$. As the other matchings contribute zero in the limit, we obtain convergence to the Wigner semi-circle as $m \to \infty$. We now prove the above assertions.

Lemma 3.1 *The limit of the average of the* $2k$*th moment of the symmetric* m*-block Toeplitz ensemble equals*

$$M_{2k;m} = C_k + \sum_{g=1}^{\lfloor \frac{k}{2} \rfloor} d(k,g) m^{-2g}, \tag{3.20}$$

where C_k *is the* k*th Catalan number and* $d(k,g) \in [0,1]$ *are constants corresponding to the total contributions from the genus* g *pairings for the* $2k$*th moment.*

Proof For the symmetric m-block Toeplitz ensemble, the analysis in Sect. 2 applies almost exactly. In the condition for $a_{ij} = a_{i'j'}$, equality replaces congruence modulo N:

- $j - i = j' - i'$ and $i \equiv i' (\mathrm{mod}\, m)$, or
- $j - i = -(j' - i')$ and $i \equiv j' (\mathrm{mod}\, m)$.

These constraints are more restrictive, so we again obtain $2^{n-l} N^{l+1}$ as an upper bound on the number of solutions. Following the previous argument, the odd moments are $M_{2k+1;m}(N) = O_k(\frac{1}{\sqrt{N}})$, and the even moments are

$$M_{2k;m}(N) = \frac{1}{N^{k+1}} \sum_{\sigma} \eta(\sigma) + O_k\left(\frac{1}{N}\right), \tag{3.21}$$

where $\eta(\sigma)$ is the number of solutions to the Diophantine equations arising from the pairings σ on $\{(1,2), (2,3), \dots, (2k,1)\}$ of the indices. Thus the odd moments vanish in the limit. Moreover, the only matchings that contribute are the ones with negative signs. To see this fact, one can follow the proof of Lemma 2.3, except working in $\mathbb{Z}$ instead of $\mathbb{Z}/N\mathbb{Z}$.

While it is known that most matchings for the real symmetric Toeplitz ensemble do not contribute fully, a general expression for the size of the contributions is unknown, though there are expressions for these in terms of volumes of Eulerian solids (see [10]) or obstructions to Diophantine equations (see [18]). These expressions imply that each matching contributes at most 1. We introduce constants to denote their contribution (this corresponds to the $m = 1$ case). This allows us to handle the real symmetric m-block Toeplitz ensemble, and (arguing as in the proof of Lemma 2.4), write the limit of the average of the $2k$th moments as

$$M_{2k;m} = \sum_{g=0}^{\lfloor \frac{k}{2} \rfloor} d(k,g) m^{-2g}. \tag{3.22}$$

Here $d(k,g)$ is the constant corresponding to the contributions of the genus g matchings. All that is left is to show that $d(k,0)$, the contributions from the non-crossing or spherical matchings, is the Catalan number C_k.

We know that the number of non-crossing matchings of $2k$ objects into k pairs is the Catalan number C_k. This is well known in the literature. Alternatively, we know the number of non-crossing matchings is $\varepsilon_0(k)$, as these are the ones that give the genus 0 sphere. The claim follows immediately from (3.2) by taking the constant term (as $g=0$) and noting $\tanh(\frac{x}{2})=\frac{x}{2}-\frac{x^3}{24}+\cdots$. We are thus reduced to proving that, even with the mod m periodicity, each of these pairings still contributes 1.

One way of doing this is by induction on matchings. Consider a non-crossing configuration of contributing matchings for the $2k$th moment. Consider an arbitrary matching in the configuration, and denote the matching by α_1. The matching corresponds to an equation $i_s-i_{s+1}=i_{t+1}-i_t$. If the matching is adjacent, meaning $s=t+1$, then i_{t+1} is free and $i_t=i_{t+2}$, and there is no "penalty" (i.e., a decrease in the contribution) from the (mod m) condition. We call this having the ends of a matching "tied" (note that adjacent matchings always tie their ends). Otherwise, note that since we are looking at even moments, there are an even number of indices. Thus, to either side of the matching α_1 there can only be an even number of indices matched between themselves, since otherwise some matching would be crossing over α_1. Thus, to either side, we are reduced to the non-crossing configurations for a lesser moment. By induction, these two sub-configurations are tied, and then trivially tie with our initial matched pair. As at each step there were no obstructions on the indices, this matching contributes fully, completing the proof. □

Our claims about convergence to semi-circular behavior now follow immediately.

Proof of Theorem 1.4(2) It is trivial to show that the symmetric m-block circulant ensemble has its limiting spectral distribution converge to the semi-ellipse as $m\to\infty$ because we have an explicit formula for its moments. From Lemma (2.4), we see that

$$\lim_{m\to\infty} M_{2k;m}(N)=\lim_{m\to\infty}\sum_{g\le\frac{k}{2}}\frac{\varepsilon_g(k)}{m^{2g}}=\varepsilon_0(k), \tag{3.23}$$

which in the proof of Lemma 3.1 we saw equals the Catalan number C_k.

We now turn to the symmetric m-block Toeplitz case. The proof proceeds similarly. From Lemma 3.1 we have

$$\lim_{m\to\infty} M_{2k;m}=\lim_{m\to\infty}\left(C_k+\sum_{g\le\frac{k}{2}}\frac{d(k,g)}{m^{2g}}\right)=C_k, \tag{3.24}$$

completing the proof. □

4 Convergence of the Limiting Spectral Measures

We investigate several types of convergence.

(1) (Almost Sure Convergence) For each k, $X_{k;m,N}\to X_{k,m}$ almost surely if

$$\mathbb{P}_m\left(\left\{A\in\Omega_m: X_{k;m,N}(A)\to X_{k,m}(A) \text{ as } N\to\infty\right\}\right)=1; \tag{4.1}$$

(2) (Convergence in Probability) For each k, $X_{k;m,N} \to X_{k,m}$ in probability if for all $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}_m\left(\left|X_{k;m,N}(A) - X_{k,m}(A)\right| > \epsilon\right) = 0; \tag{4.2}$$

(3) (Weak Convergence) For each k, $X_{k;m,N} \to X_{k,m}$ weakly if

$$\mathbb{P}_m\left(X_{k;m,N}(A) \le x\right) \to \mathbb{P}\left(X_{k,m}(A) \le x\right) \tag{4.3}$$

as $N \to \infty$ for all x at which $F_{X_{k,m}}(x) := \mathbb{P}(X_{k,m}(A) \le x)$ is continuous.

Alternate notations are to say either *with probability 1* or *strongly* for almost sure convergence and *in distribution* for weak convergence; both almost sure convergence and convergence in probability imply weak convergence. For our purposes we take $X_{k,m}$ as the random variable which is identically $M_{k,m}$, the limit of the average mth moment (i.e., $\lim_{N\to\infty} M_{k,m;N}$), which we show below exist and uniquely determine a probability distribution for our ensembles.

We have proved the first two parts of Theorem 1.4, which tells us that the limiting spectral measures exist and giving us, for the symmetric m-block circulant ensemble, a closed form expression for the density. We now prove the rest of the theorem, and determine the various types of convergence we have. We first prove the claimed uniform convergence of part (3), and then discuss the weak, in probability, and almost sure convergence of part (4).

We use characteristic functions and Fourier analysis to show uniform (and thus pointwise) convergence of the limiting spectral distribution of the symmetric m-block circulant ensemble to the semi-ellipse distribution (remember it is an semi-ellipse and not a semi-circle due to our normalization). We note that this implies L^p convergence for every p. The proof follows by showing the characteristic functions are close, and then the Fourier transform gives the densities are close.

Proof of Theorem 1.4(3) The density f_m is the Fourier transform of ϕ_m (equivalently, ϕ_m is the characteristic function associated with the density f_m, where we have to be slightly careful to keep track of the normalization of the Fourier transform; see (3.12)); similarly the Wigner distribution $f_{\mathrm{Wig}}(x)$ is the Fourier transform of ϕ, where the Wigner distribution (a semi-ellipse in our case due to our normalizations) is

$$f_{\mathrm{Wig}}(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-(\frac{x}{2})^2} & \text{if } |x| \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

As our densities are nice, we may use the Fourier inversion formula to evaluate the difference. We find for any $\epsilon > 0$ that

$$\begin{aligned} \left|\hat{\phi}_m(x) - \hat{\phi}(x)\right| &= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\phi_m(t) - \phi(t)\right)e^{-itx}\,dt\right| \\ &\le \int_{-\infty}^{\infty}\left|\phi_m(t) - \phi(t)\right|dt \\ &\ll m^{-\frac{2}{9}+\epsilon}, \end{aligned} \tag{4.5}$$

where the bound for this integral is proved in Lemma A.1 and follows from standard properties of Laguerre polynomials and Bessel functions. Thus, as $m \to \infty$, $f_m(x) = \hat{\phi}_m(x)$ converges to $f_{\text{Wig}}(x) = \hat{\phi}(x)$ for all $x \in \mathbb{R}$. As the bound on the difference depends only on m and not on x, the convergence is uniform.

We now show L^p convergence. We have L^∞ convergence because it is equivalent to a.e. uniform convergence. For $1 \le p < \infty$, we automatically have L^p convergence as we have both L^1 convergence and the L^∞ norm is bounded. □

Proof of Theorem 1.4(4) The proofs of these statements follow almost immediately from the arguments in [18, 22, 29], as those proofs relied on degree of freedom arguments. The additional structure imposed by the (mod m) relations does not substantially affect those proofs (as can seen in the generalizations of the arguments from [18] to [29] to [22]). □

5 Future Research

We discuss some natural, additional questions which we hope to study in future work.

5.1 Representation Theory

The $N \times N$ m-block circulant matrices form a semisimple algebra over $\mathbb{R}$. This algebra may be decomposed into N simple subalgebras of dimension m^2, all but one or two of which are isomorphic to $\mathrm{M}_m(\mathbb{C})$. One can show that, up to first order, this decomposition sends our measure on symmetric m-block circulant matrices to the $m \times m$ Gaussian Unitary Ensemble. One may then give a more algebraic proof of our results and circumvent the combinatorics of pairings; combining the two proofs gives a new proof of the results of [19]. This approach will appear in a more general setting in an upcoming paper of Kopp. The general result may be regarded as a Central Limit Theorem for Artin–Wedderburn decomposition of finite-dimensional semisimple algebras

5.2 Spacings

Another interesting topic to explore is the normalized spacings between adjacent eigenvalues. For many years, one of the biggest conjectures in random matrix theory was that if the entries of a full, $N \times N$ real symmetric matrix were chosen from a nice density p (say mean 0, variance 1, and finite higher moments), then as $N \to \infty$ the spacing between normalized eigenvalues converges to the scaling limit of the GOE, the Gaussian Orthogonal Ensemble (these matrices have entries chosen from Gaussians, with different variances depending on whether or not the element is on the main diagonal or not). After resisting attacks for decades, this conjecture was finally proved; see the work of Erdős, Ramirez, Schlein, and Yau [13, 14] and Tao and Vu [37, 38].

While this universality of behavior for differences seems to hold, not just for these full ensembles, but also for thin ensembles such as d-regular graphs (see the numerical observations of Jakobson, (S. D.) Miller, Rivin and Rudnick [23]), we clearly do

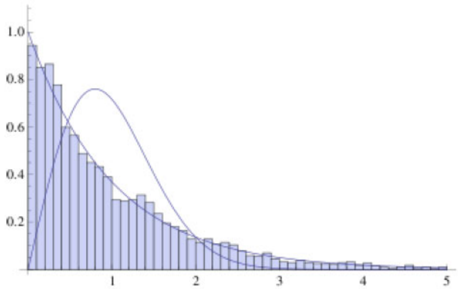


Fig. 8 Density of nonzero spacings of the 10 central eigenvalues of 100 1024 × 1024 symmetric circulant matrices, with independent entries picked i.i.d.r.v. from a Gaussian, normalized to have mean spacing 1. Compared to exponential and GOE densities

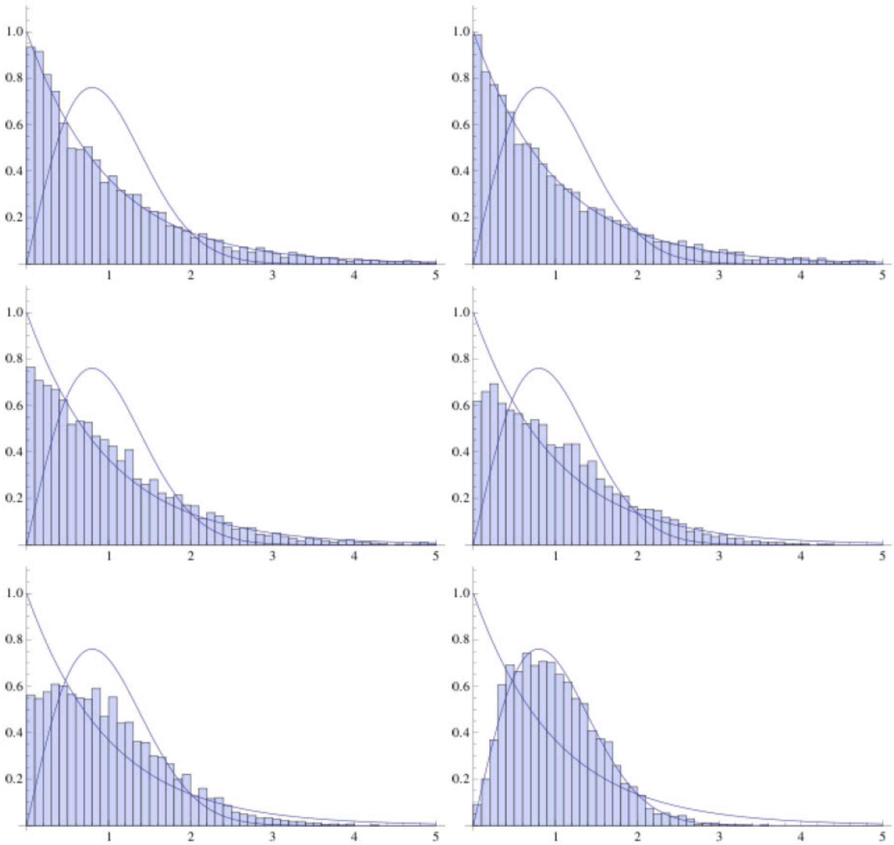


Fig. 9 Density of nonzero spacings of the 10 central eigenvalues of 100 1024 × 1024 symmetric m-block circulant matrices, with independent entries picked i.i.d.r.v. from a Gaussian, normalized to have mean spacing 1, with $m = 2, 16, 128, 256, 512, 1024$, respectively. Compared to exponential and GOE densities

not expect to see GOE behavior for all thin families. A simple counterexample are diagonal matrices; as $N \to \infty$ the density of normalized eigenvalues will be whatever density the entries are drawn from, and the spacings between normalized eigenvalues will converge to the exponential. We also see this exponential behavior in other

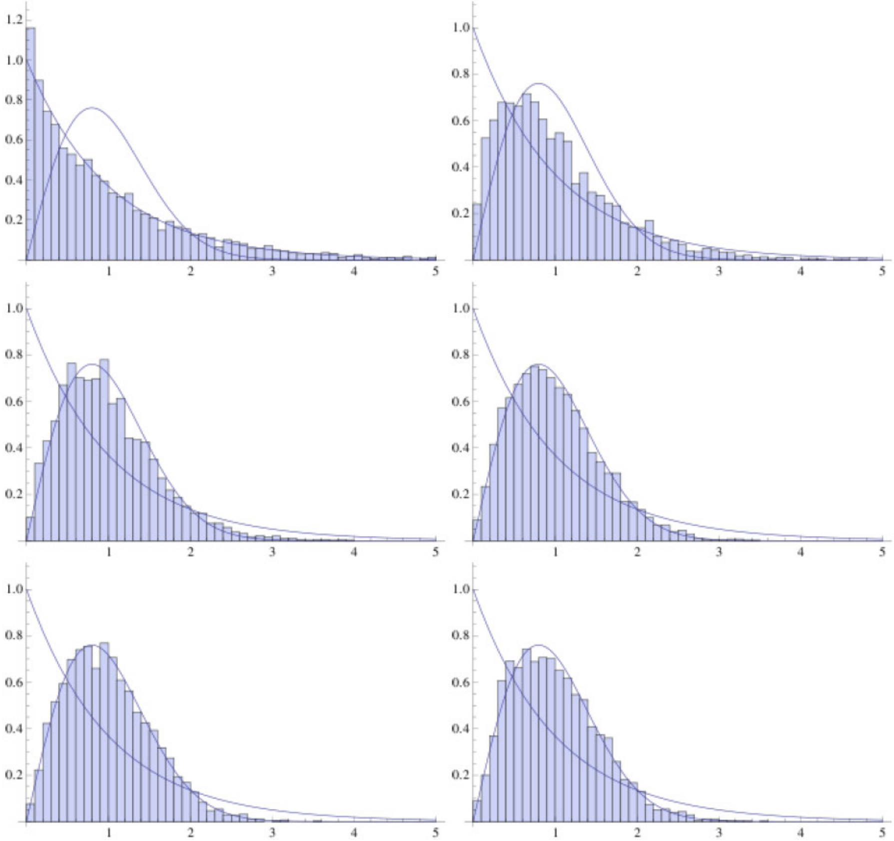


Fig. 10 Density of spacings of the 10 central eigenvalues of 100 1024×1024 symmetric m-block Toeplitz matrices, with independent entries picked i.i.d.r.v. from a Gaussian, normalized to have mean spacing 1, with $m = 1, 2, 4, 16, 128, 1024$, respectively. Compared to exponential and GOE densities

ensembles. It has numerically been observed in various Toeplitz ensembles (see [18, 29]).

For the ensemble of symmetric circulant matrices, we cannot have strictly exponential behavior because all but 1 or 2 (depending on the parity of $\frac{N}{m}$) of the eigenvalues occur with multiplicity two. This can be seen from the explicit formula for the eigenvalues of a circulant matrix. Thus, the limiting spacing density has a point of mass $\frac{1}{2}$ at 0. Nonetheless, the *nonzero* spacings appear to be distributed exponentially; see Fig. 8.

Similarly, for a symmetric m-block circulant matrix, all but $N-m$ or $N-m-1$ of the eigenvalues occur with multiplicity two. The nonzero spacings appear to have the same exponential distribution (see Fig. 9). This is somewhat surprising, given that the eigenvalue density varies with m and converges to the semi-circle as $m \to \infty$. While we see new eigenvalue densities for m constant, numerics suggest that we'll see new spacing densities for $\frac{N}{m}$ constant.

However, for symmetric m-block Toeplitz matrices, we see different behavior (see Fig. 10). The spacings look exponentially distributed for $m = 1$ and appear to converge to the GOE distribution as we increase m. In the Toeplitz case, but not in the circulant, we see the spacings behaving as the spectral densities do.

The representation theoretic approach will be used to solve the spacings problem for symmetric m-block circulant matrices in an upcoming paper of Kopp. The spacing problem for block Toeplitz matrices will require some new innovation.

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Appendix A: Pointwise Convergence as $m \to \infty$

This appendix by Gene Kopp, Steven J. Miller and Frederick Strauch.[5]

The characteristic function for the spectral measures of the period m-block circulant matrices is

$$\phi_m(t) = \frac{1}{m}\, e^{-\frac{t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left(\frac{-t^2}{m}\right)^{\ell-1}, \tag{A.1}$$

which solves the differential equation

$$t\phi_m''(t) + 3\phi_m'(t) + t\left(4 - \left(\frac{t}{m}\right)^2\right)\phi_m(t) = 0 \tag{A.2}$$

with initial condition $\phi_m(0) = 1$; letting $m \to \infty$ gives $t\phi''(t) + 3\phi'(t) + 4t\phi(t) = 0$, with initial condition $\phi(0) = 1$. The solution to the finite m equation is a Laguerre polynomial, and the $m = \infty$ limit is $\frac{J_1(2t)}{t}$ with J_1 the Bessel function of order 1.

To see this, recall that the generalized Laguerre polynomial (see [1]) has the explicit representation

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i} \frac{1}{i!}(-x)^i. \tag{A.3}$$

To compare (A.1) with (A.3), we first shift the summation index by one ($\ell \mapsto \ell + 1$) to find

$$\phi_m(t) = \frac{1}{m} e^{-\frac{t^2}{2m}} \sum_{\ell=0}^{m-1} \binom{m}{\ell+1} \frac{1}{\ell!} \left(\frac{-t^2}{m}\right)^{\ell}. \tag{A.4}$$

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Using the identity

$$\binom{m}{\ell+1} = \binom{m}{m-1-\ell} \tag{A.5}$$

we see that $n = m - 1$, $\alpha = 1$, and thus the characteristic function can be written in terms of the Laguerre polynomial:

$$\phi_m(t) = \frac{1}{m} e^{-\frac{t^2}{2m}} L_{m-1}^{(1)}\left(\frac{t^2}{m}\right), \tag{A.6}$$

or equivalently in terms of the confluent hypergeometric function

$$\phi_m(t) = e^{-\frac{t^2}{2m}} M\left(m+1, 2, -\frac{t^2}{m}\right). \tag{A.7}$$

From 13.2.2 of [1] we have $\lim_{m\to\infty} \phi_m(t) = \phi(t)$; however, we need some control on the rate of convergence.

Lemma A.1 *Let $r > \frac{1}{3}$ and $\beta = \frac{2}{3}(1-r)$. For all m and all t we have*

$$\left|\phi_m(t) - \phi(t)\right| \ll_r \begin{cases} m^{-(1-r)} & \text{if } |t| \le m^\beta, \\ t^{-\frac{3}{2}} + m^{-\frac{5}{4}} \exp(-\frac{t^2}{2m}) & \text{otherwise,} \end{cases} \tag{A.8}$$

where the implied constant is independent of m but may depend on r. This implies

$$\int_{-\infty}^{\infty} \left|\phi_m(t) - \phi(t)\right| dt \ll m^{-\frac{1-r}{3}}. \tag{A.9}$$

Letting $\epsilon > 0$ and taking $r = \frac{1}{3} + 3\epsilon$ implies the integral is $O(m^{-\frac{2}{9}+\epsilon})$.

Proof We first consider small t: $|t| \le m^\beta$ with $\beta = \frac{2}{3}(1-r)$. Using 13.3.7 of [1] with $a = m+1$, $b = 2$ and $z = -\frac{t^2}{m}$ to bound the confluent hypergeometric function M, we find

$$\begin{aligned} \phi_m(t) &= e^{-\frac{t^2}{2m}} M\left(m+1, 2, -\frac{t^2}{m}\right) \\ &= \frac{J_1(2t)}{t} + \sum_{n=1}^{\infty} A_n (2m)^{-n} (-1)^n t^{n-1} J_{n+1}(2t), \end{aligned} \tag{A.10}$$

where $A_0 = 1$, $A_1 = 0$, $A_2 = 1$ and $A_{n+1} = A_{n-1} + \frac{2m}{n+1} A_{n-2}$ for $n \ge 2$.

For any $r > \frac{1}{3}$ and m sufficiently large we have $A_n \le m^{rn}$ (we cannot do better than $r > \frac{1}{3}$ as $A_3 = \frac{2}{3}m$). This follows by induction. It is clear for $n \le 2$, and for larger n we have by the inductive assumption that

$$\begin{aligned} A_{n+1} &= A_{n-1} + \frac{2m}{n+1} A_{n-2} \leq m^{r(n-1)} + m \cdot m^{r(n-2)} \\ &= m^{r(n+1)} \cdot \left(m^{-2r} + m^{1-3r}\right); \end{aligned} \tag{A.11}$$

as $r > \frac{1}{3}$ the above is less than $m^{r(n+1)}$ for m large. If we desire a bound to hold for all m, we instead use $A_n \leq c_r m^{rn}$ for c_r sufficiently large. Substituting this bound for A_n into (A.10), noting $\frac{J_1(2t)}{t} = \phi(t)$ and using $|J_n(x)| \leq 1$ (see 9.1.60 of [1]) yields, for $|t| \leq m^{1-r}$,

$$\left|\phi_m(t) - \phi(t)\right| \leq \frac{c_r}{2m^{1-r}} \sum_{n=1}^{\infty} \left(\frac{t}{2m^{1-r}}\right)^{n-1} \ll_r m^{-(1-r)}. \tag{A.12}$$

We now turn to t large: $|t| \geq m^\beta$. Using

$$\left|\phi_m(t) - \phi(t)\right| \leq \left|\phi_m(t)\right| + \left|\phi(t)\right| \tag{A.13}$$

to trivially bound the difference, the claim follows the decay of the Bessel and Laguerre functions. Specifically (see 8.451(1) of [17]), we have $J_1(x) \ll x^{-\frac{1}{2}}$ and thus

$$\phi(t) = \frac{J_1(2t)}{t} \ll t^{-\frac{3}{2}}. \tag{A.14}$$

For $\phi_m(t)$, we use 8.978(3) of [17], which states

$$L_n^{(\alpha)}(x) = \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \cos\left(2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(n^{\frac{\alpha}{2}-\frac{3}{4}}\right), \tag{A.15}$$

so long as $\mathrm{Im}(\alpha) = 0$ and $x > 0$. Letting $x = \frac{t^2}{m}$ with $|t| \geq \frac{1}{3} \log^{\frac{1}{2}} m$, $\alpha = 1$ and $n = m - 1$ we find

$$\begin{aligned} \phi_m(t) &= m^{-1} e^{-\frac{t^2}{2m}} L_{m-1}^{(1)}\left(\frac{t^2}{m}\right) \\ &\ll m^{-1} e^{-\frac{t^2}{2m}} \left[e^{\frac{t^2}{2m}} \left(\frac{t^2}{m}\right)^{-\frac{3}{4}} m^{\frac{1}{4}} + m^{-\frac{1}{4}} \right] \\ &\ll t^{-\frac{3}{2}} + m^{-\frac{5}{4}} e^{-\frac{t^2}{2m}}. \end{aligned} \tag{A.16}$$

All that remains is to prove the claimed bound for $\int_{-\infty}^{\infty} |\phi_m(t) - \phi(t)|\, dt$. The contribution from $|t| \leq m^\beta$ is easily seen to be $O_r(\frac{m^\beta}{m^{1-r}}) = O_r(m^{-\frac{(1-r)}{3}})$ with our choice of β. For $|t| \geq m^\beta$, we have a contribution bounded by

$$\begin{aligned} 2\int_{m^\beta}^{\infty} \left(t^{-\frac{3}{2}} + m^{-\frac{5}{4}} e^{-\frac{t^2}{2m}}\right) dt &\ll m^{-\frac{\beta}{2}} + m^{-\frac{3}{4}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} \exp\left(-\frac{t^2}{2m}\right) dt \\ &\ll m^{-\frac{(1-r)}{3}} + m^{-\frac{3}{4}}, \end{aligned} \tag{A.17}$$

as the last integral is that of a Gaussian with mean zero and variance m and hence is 1. (We chose $\beta = \frac{2}{3}(1-r)$ to equalize the bounds for the two integrals.) $\square$

Appendix B: Generalized m-Block Circulant Matrices

This appendix by Steven J. Miller and Wentao Xiong.[6]

As the proofs are similar to the proof for m-block circulant matrices, we just highlight the differences. The trace expansion from before holds, as do the arguments that the odd moments vanish.

We first explore the modulo condition to compute some low moments, and show that the difference in the modulo condition between the m-block circulant matrices and the generalized m-block circulant matrices leads to different values for moments, and hence limiting spectral distributions. Thus the limiting spectral distribution depends on the frequency of each element, as well as the way the elements are arranged, in an m-pattern.

B.1 Zone-Wise Locations and Pairing Conditions

Since we have restricted the computation of moments to even moments, and have shown that the only configurations that contribute to the $2k$th moment are those in which the $2k$ matrix entries are matched in k pairs in opposite orientation, we are ready to compute the moments explicitly. We start by calculating the 2nd moment, which by (2.10) is $\frac{1}{N^2}\sum_{1\le i,j\le N} a_{ij}a_{ji}$. As long as the matrix is symmetric, $a_{ij} = a_{ji}$ and the 2nd moment is 1. We now describe the conditions for two entries $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ to be paired, denoted as $a_{i_s i_{s+1}} = a_{i_t i_{t+1}} \Longleftrightarrow (s, s+1) \sim (t, t+1)$, which we need to consider in detail for the computation of higher moments. To facilitate the practice of checking pairing conditions, we divide an $N \times N$ symmetric m-block circulant matrix into 4 zones (see Fig. 11), and then reduce an entry $a_{i_s i_{s+1}}$ in the matrix to its "basic form". Write $i_\ell = m\eta_\ell + \epsilon_\ell$, where $\eta_\ell \in \{1, 2, \ldots, \frac{N}{m}\}$ and $\epsilon_\ell \in \{0, 1, \ldots, m-1\}$, we have

(1) $0 \le i_{s+1} - i_s \le \frac{N}{2} - 1 \Rightarrow a_{i_s i_{s+1}} \in$ zone 1 and $a_{i_s i_{s+1}} = a_{\epsilon_s, m(\eta_{s+1}-\eta_s)+\epsilon_{s+1}}$;
(2) $\frac{N}{2} \le i_{s+1} - i_s \le N-1 \Rightarrow a_{i_s i_{s+1}} \in$ zone 2 and $a_{i_s i_{s+1}} = a_{\epsilon_{s+1}, m(\eta_s + \frac{N}{m} - \eta_{s+1})+\epsilon_s}$;
(3) $\frac{N}{2} \le i_s - i_{s+1} \le N-1 \Rightarrow a_{i_s i_{s+1}} \in$ zone 3 and $a_{i_s i_{s+1}} = a_{\epsilon_s, m(\eta_{s+1} + \frac{N}{m} - \eta_s)+\epsilon_{s+1}}$;
(4) $0 \le i_s - i_{s+1} \le \frac{N}{2} - 1 \Rightarrow a_{i_s i_{s+1}} \in$ zone 4 and $a_{i_s i_{s+1}} = a_{\epsilon_{s+1}, m(\eta_s-\eta_{s+1})+\epsilon_s}$.

In short, $(i_{s+1} - i_s)$ determines which diagonal $a_{i_s i_{s+1}}$ is on. If $a_{i_s i_{s+1}}$ is in zone 1 or 3 (Area I), ϵ_s determines the slot of $a_{i_s i_{s+1}}$ in an m-pattern; if $a_{i_s i_{s+1}}$ is in zone 2 or 4 (Area II), ϵ_{s+1} determines the slot of $a_{i_s i_{s+1}}$ in an m-pattern.

Recall the two basic pairing conditions, the diagonal condition that we have explored before, and the modulo condition, for which we will define an equivalence relation $\mathcal{R}$. For a real symmetric m-block circulant matrix following a generalized m-pattern and any two entries $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ in the matrix, suppose that i_s and i_{t+1}

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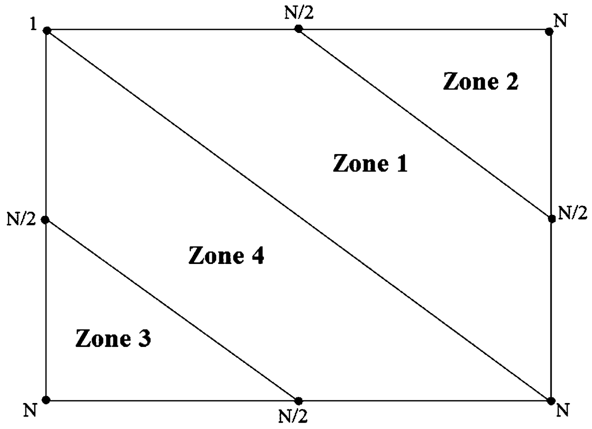

Fig. 11 The four zones for m-block circulant matrices

are the indices that determine the slot of the respective entries, then $i_s \mathcal{R} i_{t+1}$ if and only if $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are in certain slots in an m-pattern such that these two entries can be equal. For example, for the $\{a, b\}$ pattern, $i_s \mathcal{R} i_{t+1} \Longleftrightarrow i_s \equiv i_{t+1} (\bmod 2)$; for the $\{a, a, b, b\}$ pattern, $i_s \mathcal{R} i_{t+1} \Longleftrightarrow \mod(i_s, 4), \quad \mod(i_{t+1}, 4) \in \{1, 2\}$ or $\mod(i_s, 4), \quad \mod(i_{t+1}, 4) \in \{3, 0\}$.

We now formally define the two pairing conditions.

(1) (diagonal condition) $i_s - i_{s+1} \equiv -(i_t - i_{t+1}) \pmod N$.
(2) (modulo condition) $i_s \mathcal{R} i_{t+1}$ or $i_{s+1} \mathcal{R} i_t$, depending on which zone(s) $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are located in.

Since the diagonal condition implies a Diophantine equation for each of the k pairs of matrix entries, we only need to choose $k+1$ out of $2k$ i_ℓ's, and the remaining i_ℓ's are determined. This shows that, trivially, the number of non-trivial configurations is bounded above by N^{k+1}. In addition, the diagonal condition always ensure that $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are located in different areas. For instance, if $a_{i_s i_{s+1}} \in$ zone 1 and $i_s - i_{s+1} = -(i_t - i_{t+1})$, then $a_{i_s i_{s+1}} \in$ zone 4; if $a_{i_s i_{s+1}} \in$ zone 1 and $i_s - i_{s+1} = -(i_t - i_{t+1}) - N$, then $a_{i_s i_{s+1}} \in$ zone 2, etc. Thus, if i_s determines the slot for $a_{i_s i_{s+1}}$ in an m pattern, then i_{t+1} determines for $a_{i_t i_{t+1}}$; if i_{s+1} determines the slot for $a_{i_s i_{s+1}}$, then i_t determines for $a_{i_t i_{t+1}}$, and vice versa.

Considering the "basic" form of the entries, the two conditions above are equivalent to

(1) (diagonal condition) $(m\eta_s + \epsilon_s) - (m\eta_{s+1} + \epsilon_{s+1}) \equiv -(m\eta_t + \epsilon_t) + (m\eta_{t+1} + \epsilon_{t+1}) (\bmod N) \Rightarrow m(\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1}) + (\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = 0$ or $\pm N$.
(2) (modulo condition) $\epsilon_s \mathcal{R} \epsilon_{t+1}$ or $\epsilon_{s+1} \mathcal{R} \epsilon_t$.

Since $m|N$, this requires $m|(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$. Given the range of the η_ℓ's and ϵ_ℓ's, we have $\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1} = 0$ or $\pm m$, which indicates that

$$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = 0, \pm 1, \frac{N}{m}, \frac{N}{m} \pm 1, -\frac{N}{m}, \text{ or } -\frac{N}{m} \pm 1. \tag{B.1}$$

As discussed before, if we allow repeated elements in an m-pattern, the equivalence relation $\mathcal{R}$ no longer necessitates a congruence relation as in pattern where each element is distinct. While the computation of high moments for general m-patterns appears intractable, fortunately we are able to illustrate how the difference in the modulo condition affects moment values by comparing the low moments for two simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$.

B.2 The Fourth Moment

Although we can show that the higher moments differ by the way the elements are arranged in an m-pattern, the 4th moment is in fact independent of the arrangement of elements. We show that the 4th moment for any m-pattern is determined solely by the frequency at which each element appears, and refer the reader to Appendix B.3 of [25] (or [45]) for the computation that the 6th moment depends on not just the frequencies but also the pattern; we omit the proof as it is similar to the computation of the 4th moment, although significantly more book-keeping is required. Briefly, for the higher moments for patterns with repeated elements, there exist "obstructions to modulo equations" that make trivial some non-trivial configurations for patterns without repeated elements. Due to the obstructions to modulo equations, some configurations that are non-trivial for all-distinct patterns become trivial for patterns with repeated elements, making the higher moments for repeated patterns smaller.

Lemma B.1 *For an ensemble of real symmetric period m-block circulant matrices of size N, if within each m-pattern we have n i.i.d.r.v. $\{\alpha_r\}_{r=1}^n$, each of which has a fixed number of occurrences ν_r such that $\sum_{r=1}^n \nu_r = m$, the 4th moment of the limiting spectral distribution is* $2 + \sum_{r=1}^n (\frac{\nu_r}{m})^3$.

By (2.10), we calculate $\frac{1}{N^{\frac{4}{2}+1}} \sum_{1\le i,j,k,l\le N} a_{ij}a_{jk}a_{kl}a_{li}$ for the 4th moment. There are 2 ways of matching the 4 entries in 2 pairs:

(1) (adjacent, 2 variations) $a_{ij} = a_{jk}$ and $a_{kl} = a_{li}$ (or equivalently $a_{ij} = a_{li}$ and $a_{jk} = a_{kl}$);
(2) (diagonal, 1 variation) $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$.

There are 3 matchings, with the two adjacent matchings contributing the same to the 4th moment. We first consider one of the adjacent matchings, $a_{ij} = a_{jk}$ and $a_{kl} = a_{li}$. The pairing conditions (B.1) in this case are:

(1) (diagonal condition) $i - j \equiv k - j (\text{mod } N)$, $k - l \equiv i - l (\text{mod } N)$;
(2) (modulo condition) $i\mathcal{R}k$ or $j\mathcal{R}j$, $k\mathcal{R}i$ or $l\mathcal{R}l$.

Since $1 \le i, j, k, l \le N$, the diagonal condition requires $i = k$, and then the modulo condition follows trivially, regardless of the m-pattern we study. Hence, we can choose j and l freely, each with N choices, i freely with N choices, and then k is fixed. This matching then contributes $\frac{N^3}{N^{\frac{4}{2}+1}} = 1$ (fully) to the 4th moment, so does the other adjacent matching.

We proceed to the diagonal matching, $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$. The pairing conditions (B.1) in this case are:

(1) (diagonal condition) $i - j \equiv l - k (\text{mod } N)$, $j - k \equiv i - l (\text{mod } N)$;

(2) (modulo condition) $i\mathcal{R}l$ or $j\mathcal{R}k$, $j\mathcal{R}i$ or $k\mathcal{R}l$.

The diagonal condition $j-k \equiv i-l \pmod N$ is equivalent to $i-j \equiv l-k \pmod N$, which entails

(1) $i+k=j+l$, or
(2) $i+k=j+l+N$, or
(3) $i+k=j+l-N$.

In any case, we only need to choose 3 indices out of i, j, l, k, and then the last one is fixed. In the following argument, without loss of generality, we choose (i, j, l) and thus fix k.

For a general m-pattern, we write $i=4\eta_1+\epsilon_1$, $j=4\eta_2+\epsilon_2$, $k=4\eta_3+\epsilon_3$, $l=4\eta_4+\epsilon_4$, where $\eta_1,\eta_2,\eta_3,\eta_4 \in \{0,1,\dots,\frac{N}{m}\}$ and $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{0,1,\dots,m-1\}$. Before we consider the ϵ_ℓ's, we note that there exist Diophantine constraints. For example, if $i+k=j+l$, given that $1\le i,j,l\le N$, $k=j+l-i$ also needs to satisfy $1\le k\le N$. As a result, we need $0\le \eta_2+\eta_4-\eta_1\le \frac{N}{4}$. Note that, due to the ϵ_ℓ's, sometimes we may have $0\le \eta_2+\eta_4-\eta_1\le \frac{N}{4}+\varepsilon$, where the error term $\varepsilon\in(-\frac{m}{2},\frac{m}{2})$ and only trivially affects the number of choices of (η_2,η_4,η_1) for a fixed m as $N\to\infty$.

We now explore the Diophantine constraints for each variation of the diagonal condition (B.1). The $i+k=j+l$ case is similar to that in [18], where, in a Toeplitz matrix, the diagonal condition only entails $i+k=j+l$, and there are obstructions to the system of Diophantine equations following the diagonal condition. However, the circulant structure that adds $i+k=j+l+N$ and $i+k=j+l-N$ to the diagonal condition fully makes up the Diophantine obstructions. This explains why the limiting spectral distribution for ensembles of circulant matrices has the moments of a Gaussian, while that for ensembles of Toeplitz matrices has smaller even moments. We now study the 3 possibilities of the diagonal condition for the circulant structure.

(1) Consider $i+k=j+l$. We use Lemma 2.5 from [18] to handle the obstructions to Diophantine equations, which says: *Let* $I_N=\{1,\dots,N\}$. *Then* $\#\{x,y,z\in I_N : 1\le x+y-z\le N\}=\frac{2}{3}N^3+\frac{1}{3}N$.

   In our case, let $M=\frac{N}{m}$. The number of possible combinations of (η_2,η_4,η_1) that allow $0\le\eta_3\le\frac{N}{4}$ is $\frac{2}{3}M^3+\frac{1}{3}M$.[7] For each of η_2,η_4,η_1, we have m free choices of ϵ_ℓ, and thus the number of (i,j,l) is $m^3(\frac{2}{3}M^3+\frac{1}{3}M)=\frac{2}{3}N^3+O(N)$.

(2) Consider $i+k=j+l+N$. Note $1\le k\le N$ requires $0\le\eta_2+\eta_4-\eta_1+\frac{N}{m}\le\frac{N}{m}\Rightarrow -\frac{N}{m}\le\eta_2+\eta_4-\eta_1\le 0$. Similar to the $i+k=j+l$ case, we write $M=\frac{N}{m}$ and $S=\eta_2+\eta_4$, and then $-\frac{N}{m}\le S-\eta_1\le 0\Rightarrow S\le\eta_1\le M+S$ where obviously $S\le M$. We have $S+1$ ways to choose (η_2,η_4) s.t. $\eta_2+\eta_4=S$, and $M-S+1$ choices of η_1. The number of (i,j,l) is thus

$$m^3\sum_{S=0}^{M}(S+1)(M-S+1)=m^3\left(\frac{M^3}{6}+M^2+\frac{5}{6}M\right)=\frac{N^3}{6}+O\big(N^2\big). \quad \text{(B.2)}$$

[7] In [18], the related lemma is proven for $\eta_2,\eta_4,\eta_1\in\mathbb{N}_+$, i.e., no cases where $\eta_2\eta_4\eta_1=0$. Thus we are supposed to start from $S=0$; however, as $N\to\infty$, the error from this becomes negligible.

(3) Consider $i+k=j+l-N$. Now $1\le k\le N$ requires $0\le \eta_2+\eta_4-\eta_1-\frac{N}{m}\le \frac{N}{m}\Rightarrow \frac{N}{m}\le \eta_2+\eta_4-\eta_1\le \frac{2N}{m}$. Again, we write $M=\frac{N}{m}$ and $S=\eta_1+\eta_4$, and then $M\le S-\eta_1\le 2M\Rightarrow S-2M\le \eta_1\le S-M$ where obviously $S\ge M$. We have $2M-S+1$ ways to choose (η_2,η_4) s.t. $\eta_2+\eta_4=S$, and $S-M+1$ choices of η_1. The number of (i,j,l) is thus

$$m^3\sum_{S=M}^{2M}(2M-S+1)(S-M+1)=m^3\left(\frac{M^3}{6}+M^2+\frac{5}{6}M\right)$$
$$=\frac{N^3}{6}+O\bigl(N^2\bigr). \tag{B.3}$$

Therefore, with the additional diagonal conditions $i+k=j+l+N$ and $i+k=j+l-N$ induced by the circulant structure, the number of (i,j,l) is of the order $(\frac{2}{3}+\frac{1}{6}+\frac{1}{6})N^3=N^3$, i.e. the circulant structure makes up the obstructions to Diophantine equations in the Toeplitz case. Since the η_ℓ's do not matter for the modulo condition, to make a non-trivial configuration, we may choose three η_ℓ's freely, each with $\frac{N}{m}$ choices, and then choose some ϵ_ℓ's that satisfy the modulo condition, which we will study below.

For the modulo condition, it is necessary to figure out which zones the four entries are located in. Recall that the diagonal condition will always ensure that two paired entries are located in different areas. For the 4th moment, each of the 3 variations of the diagonal condition is sufficient to ensure that any pair of entries involved are located in the right zones. We may check this rigorously by enumerating all possibilities of the zone-wise locations of the 4 entries, e.g. if $i+k=j+l+N$, if $a_{ij}\in$ zone 1, then $a_{kl}\in$ zone 2.[8] As a result, for a pair of matrix elements in the diagonal matching, say $a_{ij}=a_{kl}$, if i determines the slot in an m-pattern for a_{ij} and thus matters for the modulo condition, then l determines for a_{kl}; if j determines for a_{ij}, then k determines for a_{kl}, and vice versa.

With the zone-wise issues settled, we study how to obtain a non-trivial configuration for the 4th moment. Recall the modulo condition for the diagonal matching: $i\mathcal{R}l$ or $j\mathcal{R}k$, $j\mathcal{R}i$ or $k\mathcal{R}l$. This entails $2^2=4$ sets of equivalence relations,

$$i\mathcal{R}l\mathcal{R}j;\, i\mathcal{R}l\mathcal{R}k,\, j\mathcal{R}k\mathcal{R}i,\, j\mathcal{R}k\mathcal{R}l. \tag{B.4}$$

Each set of equivalence relations appears with a certain probability, depending on the zone-wise locations of the 4 entries. For example, $i\mathcal{R}l\mathcal{R}j$ follows from $i\mathcal{R}l$ and $j\mathcal{R}i$, which requires both a_{ij} and $a_{jk}\in$ Area I. Regardless of the probability with which each set occurs, we choose one free index with N choices, and then another two indices such that these 3 indices are related to each other under $\mathcal{R}$. The number of choices of the two indices after the free one is determined solely by the number of occurrences of the elements in an m-pattern.

[8] This enumeration is complicated since the zone where an entry a_{ij} is located imposes restrictions on the choice of i,j, e.g. when $a_{i,j}\in$ zone 2, we have $i\ge\frac{N}{2}$ and $j\le\frac{N}{2}$.

We give a specific example of making a non-trivial configuration for the 4th for two simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$. Under the condition $i + k = j + l$, if $a_{ij} \in$ zone 1 and $a_{jk} \in$ zone 3, then $a_{kl} \in$ zone 4 and $a_{li} \in$ zone 2. We first select η_1, η_2, η_4 such that i, j, l and $k = j + l - i$ satisfy the zone-wise locations.[9] In this case, based on pairing conditions (B.1), pairing $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$ will require $\epsilon_1 \mathcal{R} \epsilon_4$ and $\epsilon_2 \mathcal{R} \epsilon_1$, or equivalently $\epsilon_1 \mathcal{R} \epsilon_2 \mathcal{R} \epsilon_4$. Without loss of generality, we can start with a free ϵ_1 with 4 choices, then there are 2 free choices for each of ϵ_2 and ϵ_4, and then we have a non-trivial configuration. We have similar stories under the other two variations of the diagonal condition and with other zone-wise locations of a_{ij} and a_{kl}. Therefore, we can choose three out of four η_ℓ's freely, each with $\frac{N}{4}$ choices, then one ϵ_ℓ with 4 choices, then another two ϵ_ℓ's each with 2 choices, and finally the last index is determined under the diagonal condition. As discussed before, such a choice of indices will always satisfy the zone-wise requirements and thus the ϵ-based pairing conditions. Thus there are $(\frac{N}{4})^3 \cdot 4 \cdot 2 \cdot 2 = \frac{N^3}{4}$ choices of (i, j, k, l) that will produce a non-trivial configuration. It follows that the contribution from the diagonal matching to the 4th moment is $\frac{1}{N^3}(\frac{2}{3} + \frac{1}{6} + \frac{1}{6})\frac{N^3}{4} = \frac{1}{4}$.

The computation of the 4th moment for the simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$ can be immediately generalized to the 4th moment for other patterns. As emphasized before, both adjacent matchings contribute fully to the 4th moment regardless of the m-pattern. For diagonal matching, the system of Diophantine equations induced by the diagonal condition are also independent of the m-pattern in question, and the way we count possible configurations can be easily generalized to an arbitrary m-pattern. We have thus proved Lemma B.1.

Note that Lemma B.1 implies that the 4th moment for any pattern depends solely on the frequency at which each element appears in an m-period. Besides the $\{a, a, b, b\}$ pattern that we have studied in depth, we may easily test two extreme cases. One case where $n = m$, i.e. each random variable appears only once, represents the m-block circulant matrices from Theorem 1.4 for which the 4th moment is $2 + \frac{1}{m^2}$ (and $m = 1$ represents the circulant matrices for which the 4th moment is 3). Numerical simulations for numerous patterns including $\{a, a, b\}$, $\{a, b, b\}$, $\{a, b, b, a\}$, $\{a, b, c, a, b, c\}$, $\{a, b, c, d, e, e, d, c, b, a\}$ et cetera support Lemma B.1 as well; we present results of some simulations in Tables 1, 2 and 3.

B.3 Existence and Convergence of High Moments

Although it is impractical to find every moment for a general m-block circulant pattern using brute-force computation, we are still able to prove that, for any m-block circulant pattern, every moment exists, is finite (and satisfies certain bounds), and that there exists a limiting spectral distribution. In addition, the empirical spectral measure of a typical real symmetric m-block circulant matrix converge to this limiting measure, and we have convergence in probability and almost sure convergence.

We have shown that all the odd moments vanish as $N \to \infty$, and thus we focus on the even moments. We need to prove the following theorem.

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[9] It is noteworthy that the specific location of an element still depends on the ϵ_ℓ's, but as $N \to \infty$, the probability that the η_ℓ's alone determine the zone-wise locations of elements approaches 1, i.e. the probability that adding the ϵ_ℓ's changes the zone-wise location of an element approaches 0.

Table 1 Comparison of moments for various patterns involving a and b. The first column are the theoretical values for the moments of the pattern a, b, and the final are the moments of the standard normal. The middle three columns are 200 simulations of 4000×4000 matrices

k	*abab* (theory)	*abab* (observed)	*aabb* (observed)	*abba* (observed)	$N(0,1)$
2	1.0000	1.0016	1.0014	0.9972	1
4	2.2500	2.2583	2.2541	2.2405	3
6	7.5000	7.5577	7.3212	7.2938	15
8	32.8125	33.2506	30.4822	30.5631	105
10	177.1880	180.8270	153.9530	155.6930	945

Table 2 Comparison of moments for various patterns involving a and b. The first column are the theoretical values for the moments of the pattern a, b. The remaining columns are 200 simulations of 3600×3600 matrices

k	*ababab*	*aaabbb*	*aaaabbbb*	*aaaaabbbbb*	*aababb*
2	1.0000	1.0008	1.0001	0.9984	0.9996
4	2.2500	2.2541	2.2441	2.2449	2.2502
6	7.5000	7.3011	7.2098	7.2551	7.2319
8	32.8125	30.3744	29.5004	30.0127	29.5378
10	177.1880	155.0380	145.8240	150.7220	145.4910

Table 3 Comparison of moments for various patterns involving a and b. The first column are the theoretical values for the moments of the pattern a, b, c. The remaining columns are 200 simulations of 3600×3600 matrices

k	*abcabc*	*abccba*	*aabbcc*	*abbcca*	*aabcbc*
2	1.0000	1.0005	1.0006	0.9983	1.0013
4	2.1111	2.1122	2.1153	2.1047	2.1161
6	6.1111	6.0248	6.0540	6.0083	6.0235
8	22.0370	20.9398	21.2004	20.9908	20.8411
10	94.6296	85.0241	87.0857	85.9902	84.2097

Theorem B.2 *For any patterned m-block circulant matrix ensemble,* $\lim_{N\to\infty} M_{2k}(N)$ *exists and is finite.*

Proof It is trivial that $M_{2k}(N)$ is finite. As discussed before, it is bounded below by the $2k$th moment for the ensemble of m-block circulant matrices where, in the m-pattern, each element is distinct, and more importantly it is bounded above by the $2k$th moment for the ensemble of circulant matrices, and we know that the limiting spectral distribution for this matrix ensemble is a Gaussian.

We now show that $\lim_{N\to\infty} M_{2k}(N)$ exists. To calculate $M_{2k}(N)$, we match $2k$ elements from the matrix, $\{a_{i_1 i_2}, a_{i_2 i_3}, \ldots, a_{i_{2k} i_1}\}$, in k pairs, where $i_\ell \in \{1, 2, \ldots, N\}$ and this will give $(2k-1)!!$ matchings. For each matching, there are a certain number of configurations, and most of such configurations do not contribute to the moments as $N \to \infty$.

For the m-block circulant pattern, the equivalence relation $\mathcal{R}$ implies that $\epsilon_s \mathcal{R} \epsilon_{t+1} \Leftrightarrow \epsilon_s = \epsilon_{t+1}$, and since $m|(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$, we have $\epsilon_{s+1} = \epsilon_t$ as well (see (B.1)).[10] Thus $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = 0$ or $\pm \frac{N}{m}$, three equations that have $(\frac{N}{m})^3 + O((\frac{N}{m})^2)$ solutions in total, as we have shown in the 4th moment computation.

However, if there are repeated elements in an m-period, then $\epsilon_s \mathcal{R} \epsilon_{t+1}$ no longer necessitates $\epsilon_s = \epsilon_{t+1}$, and it is possible that $(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = \pm m$. Thus, the zone-wise locations of elements matter in making non-trivial configurations. Recall that the zone-wise location (see (B.1)) of an element $a_{i_s i_{s+1}}$ is determined by $(i_{s+1} - i_s)$: if $a_{i_s i_{s+1}}$ is in zone 1 or 3 (Area I), ϵ_s determines the slot of $a_{i_s i_{s+1}}$ in an m-period; if $a_{i_s i_{s+1}}$ is in zone 2 or 4 (Area II), ϵ_{s+1} determines the slot of $a_{i_s i_{s+1}}$ in an m-period. In addition, the diagonal condition will always ensure that two paired entries $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are located in different areas.

Recall that for any matching $\mathcal{M}$, the k pairs of matrix elements, each pair in the form of $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, are fixed. For any $\mathcal{M}$, to make a non-trivial configuration, we first choose an ϵ vector of length $2k$. If we choose all the ϵ_ℓ's freely, there are m^{2k} possible choices for an ϵ vector, most of which do not meet the modulo condition, and trivially, m^{2k} is an upper bound for the number of valid ϵ vectors. It is noteworthy that out of the $2k$ ϵ_ℓ's of an ϵ vector, only some of the ϵ_ℓ's will matter for the modulo condition. Which ϵ_ℓ's in fact matter depends on how we pair the $2k$ matrix entries $a_{i_s i_{s+1}}$'s and the zone-wise locations of the paired $a_{i_s i_{s+1}}$'s, which we cannot determine without fixing the η_ℓ's (and thus the i_ℓ's).

However, for any matching, the way we pair the $2k$ matrix entries into k pairs is fixed, and for each fixed pair $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, two ϵ_ℓ's will matter for the modulo condition: either $\epsilon_s \mathcal{R} \epsilon_{t+1}$ or $\epsilon_{s+1} \mathcal{R} \epsilon_t$. Thus there are 2^k ways to choose k pairs of ϵ_ℓ's for each matching. For each way of fixing the k pairs of ϵ_ℓ's, we examine each ϵ pair, say $(\epsilon_{\ell_1}, \epsilon_{\ell_2})$, and there are a certain number of choices of $(\epsilon_{\ell_1}, \epsilon_{\ell_2})$ such that $\epsilon_{\ell_1} \mathcal{R} \epsilon_{\ell_2}$. Continuing in this way, for each ϵ pair, we choose two ϵ_ℓ's that satisfy the equivalence relation $\mathcal{R}$. Note that an ϵ_ℓ may matter twice, once, or never for the modulo condition depending on the zone-wise locations of the $a_{i_s i_{s+1}}$'s. We then choose the other ϵ_ℓ's that do not matter for the modulo condition such that for each pair of $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, we have $\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1} = 0$ or $\pm m$, and finally we have a valid ϵ vector. The number of valid ϵ vectors will be determined by m, k, and the pattern of an m-period, but will be independent of N since the system of k equivalence relations for the modulo condition does not involve N.

With a valid ϵ vector, we have fixed the zone-wise locations of the $2k$ matrix elements by fixing the ϵ_ℓ's that matter for the modulo condition. We now turn to the diagonal condition and study the η_ℓ's. With k equations in the form of

$$m(\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1}) + (\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = 0 \quad \text{or } \pm N, \tag{B.5}$$

[10]This explains why, for an m-pattern without repeated elements, the zone-wise locations of matrix entries do not matter in making a non-trivial configuration.

and $(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$ known in each of the k equations, we in fact have k equations in the form of

$$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma, \tag{B.6}$$

where $\gamma \in \{0, \pm 1, \frac{N}{m}, \frac{N}{m} \pm 1, -\frac{N}{m}, -\frac{N}{m} \pm 1\}$. This gives us $k+1$ degrees of freedom in choosing the η_ℓ's, and trivially, we can have at most $(\frac{N}{m})^{k+1}$ vectors of η_ℓ's. Since the ϵ vector is fixed, for one equation $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, there are only 3 choices of γ. With k equations in this form, we have at most 3^k systems of η equations. Note that not all of the η vectors satisfying an η equation system derived from the diagonal condition will help make a non-trivial configuration, since the η_ℓ's need to be chosen such that the resulted $a_{i_s i_{s+1}}$'s will satisfy the zone-wise locations in order to be coherent with the pre-determined ϵ vector. For example, if in a pair of matrix entries $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$ where $\epsilon_s \mathcal{R} \epsilon_{t+1}$, even though the η_ℓ's are chosen such that $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, it is possible that $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are located in certain zones such that we need $\epsilon_{s+1} \mathcal{R} \epsilon_t$ to ensure a non-trivial configuration.

The following steps mirror those in [18]. Denote an η equation system by $\mathcal{S}$. For any $\mathcal{S}$ we have k equations with $\eta_1, \eta_2, \ldots, \eta_{2k} \in \{1, 2, \ldots, \frac{N}{m}\}$. Let $z_\ell = \frac{\eta_\ell}{N/m} \in \{\frac{m}{N}, \frac{2m}{N}, \ldots, 1\}$. Without the zone-wise concerns discussed before, the system of k equations would have $k+1$ degrees of freedom and determine a nice region in the $(k+1)$-dimensional unit cube. Taking into account the zone-wise concerns, however, we will still have $k+1$ degrees of freedom. For example, for a pair of matrix elements $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, the system $\mathcal{S}$ requires $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$. If we need $\epsilon_s \mathcal{R} \epsilon_{t+1}$ to make a non-trivial configuration, say $a_{i_s i_{s+1}} \in$ zone 1, then we will obtain an additional equation $0 \le i_{s+1} - i_s \le \frac{N}{2} - 1 \Rightarrow 0 \le (\eta_{s+1} - \eta_s) + \epsilon_{s+1} - \epsilon_s \le \frac{N}{2} - 1$ with $(\epsilon_{s+1} - \epsilon_s) \in \{-m+1, -m+2, \ldots, 0, 1, \ldots, m-2, m-1\}$. Based on the region determined by $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, this additional zone-related restriction will only allow a slice of the region for us to choose valid η_ℓ's. With k zone-wise restrictions, only a proportion of the original region in the unit cube will be preserved for the choice of the η vector. Nevertheless, the "width" of each slice is of order $\frac{N}{2}$, and we still have $k+1$ degrees of freedom.

Therefore, with m fixed and as $N \to \infty$, we obtain to first order the volume of this region, which is finite. Unfolding back to the η_ℓ's, we obtain $M_{2k}(\mathcal{S})(\frac{N}{m})^{k+1} + O_k((\frac{N}{m})^k)$, where $M_{2k}(\mathcal{S})$ is the volume associated with this η system. Summing over all η systems, we obtain the number of non-trivial configurations for the $2k$th moment from this particular ϵ vector. Next, within a given matching $\mathcal{M}$, we sum over all valid ϵ vectors, the number of which is independent of N as we have shown before. In the end, we sum over the $(2k-1)!!$ matchings to obtain $M_{2k} N^{k+1} + O_k(N^k)$, and the $2k$th moment is simply $\frac{M_{2k} N^{k+1} + O_k(N^k)}{N^{k+1}} = M_{2k} + O(\frac{1}{N})$. □

The above proves the existence of the moments. The convergence proof follows with only minor changes to the convergence proofs from [18, 29].

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1965). Online at http://people.math.sfu.ca/~cbm/aands/
2. Baik, J., Borodin, A., Deift, P., Suidan, T.: A model for the bus system in Cuernevaca (Mexico). Math. Phys. (2005), 1–9. Online available at http://arxiv.org/abs/math/0510414
3. Basak, A., Bose, A.: Limiting spectral distribution of some band matrices. Periodica Math. Hungarica **63**(1), 113–150 (2011)
4. Basak, A., Bose, A.: Balanced random Toeplitz and Hankel matrices. Electron. Comm. Probab. **15**, 134–148
5. Banerjee, S., Bose, A.: Noncrossing partitions, Catalan words and the semicircle law. J. Theor. Probab. doi:10.1007/s10959-011-0365-4
6. Beckwith, O., Miller, S.J., Shen, K.: Distribution of eigenvalues of weighted, structured matrix ensembles, preprint
7. Bose, A., Chatterjee, S., Gangopadhyay, S.: Limiting spectral distributions of large dimensional random matrices. J. Indian Stat. Assoc. **41**, 221–259 (2003)
8. Bose, A., Hazra, R.S., Saha, K.: Patterned random matrices and notions of independence, Technical Report R3/2010 (2010), Stat-Math Unit, Kolkata. Available online at http://www.isical.ac.in/~statmath
9. Bose, A., Mitra, J.: Limiting spectral distribution of a special circulant. Stat. Probab. Lett. **60**(1), 111–120 (2002)
10. Bryc, W., Dembo, A., Jiang, T.: Spectral measure of large random Hankel, Markov, and Toeplitz matrices. Ann. Probab. **34**(1), 1–38 (2006)
11. Dyson, F.: Statistical theory of the energy levels of complex systems: I, II, III. J. Math. Phys. **3**, 140–156 (1962), 157–165, 166–175
12. Dyson, F.: The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics. J. Math. Phys. **3**, 1199–1215 (1962)
13. Erdős, L., Ramirez, J.A., Schlein, B., Yau, H.-T.: Bulk Universality for Wigner Matrices, preprint. http://arxiv.org/abs/0905.4176
14. Erdős, L., Schlein, B., Yau, H.-T.: Wegner estimate and level repulsion for Wigner random matrices, preprint. http://arxiv.org/abs/0905.4176
15. Firk, F.W.K., Miller, S.J.: Nuclei, primes and the random matrix connection. Symmetry **1**, 64–105 (2009). doi:10.3390/sym1010064
16. Forrester, P.J.: Log-Gases and Random Matrices. London Mathematical Society Monographs, vol. 34. Princeton University Press, Princeton (2010)
17. Gradshteyn, I., Ryzhik, I.: Tables of Integrals, Series, and Products. Academic Press, New York (1965)
18. Hammond, C., Miller, S.J.: Eigenvalue spacing distribution for the ensemble of real symmetric Toeplitz matrices. J. Theor. Probab. **18**(3), 537–566 (2005)
19. Harer, J., Zagier, D.: The Euler characteristic of the moduli space of curves. Invent. Math. **85**, 457–485 (1986)
20. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
21. Hayes, B.: The spectrum of Riemannium. Am. Sci. **91**(4), 296–300 (2003)
22. Jackson, S., Miller, S.J., Pham, V.: Distribution of eigenvalues of highly palindromic Toeplitz matrices. http://arxiv.org/abs/1003.2010
23. Jakobson, D., Miller, S.D., Rivin, I., Rudnick, Z.: Eigenvalue spacings for regular graphs. In: Emerging Applications of Number Theory, Minneapolis, 1996. The IMA Volumes in Mathematics and its Applications, vol. 109, pp. 317–327. Springer, New York (1999)
24. Kargin, V.: Spectrum of random Toeplitz matrices with band structure. Electron. Commun. Probab. **14**, 412–421 (2009)
25. Kopp, G.S., Koloğlu, M., Miller, S.J., Strauch, F., Xiong, W.: The limiting spectral measure for ensembles of symmetric block circulant matrices. arXiv version. http://arxiv.org/abs/1008.4812
26. Krbalek, M., Seba, P.: The statistical properties of the city transport in Cuernavaca (Mexico) and Random matrix ensembles. J. Phys. A, Math. Gen. **33**, L229–L234 (2000)
27. Ledoux, M.: A recursion formula for the moments of the Gaussian orthogonal ensemble. Ann. IHP **45**(3), 754–769 (2009)
28. Liu, D.-Z., Wang, Z.-D.: Limit distribution of eigenvalues for random Hankel and Toeplitz band matrices, to appear in J Theor Probab. http://arxiv.org/abs/0904.2958

29. Massey, A., Miller, S.J., Sinsheimer, J.: Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices. J. Theor. Probab. **20**(3), 637–662 (2007)
30. Meckes, M.: The spectra of random abelian G-circulant matrices, preprint. http://arxiv.org/PS_cache/arxiv/pdf/1109/1109.5675v1.pdf
31. Miller, S.J., Novikoff, T., Sabelli, A.: The distribution of the second largest eigenvalue in families of random regular graphs. Exp. Math. **17**(2), 231–244 (2008)
32. McKay, B.: The expected eigenvalue distribution of a large regular graph. Linear Algebra Appl. **40**, 203–216 (1981)
33. Montgomery, H.: The pair correlation of zeros of the zeta function. In: Analytic Number Theory. Proceedings of Symposia in Pure Mathematics, vol. 24, pp. 181–193. AMS, Providence (1973)
34. Stein, E., Shakarchi, R.: Fourier Analysis: An Introduction. Princeton University Press, Princeton (2003)
35. Stein, E., Shakarchi, R.: Complex Analysis. Princeton University Press, Princeton (2003)
36. Takacs, L.: A moment convergence theorem. Am. Math. Mon. **98**(8), 742–746 (1991)
37. Tao, T., Vu, V.: From the Littlewood–Offord problem to the circular law: Universality of the spectral distribution of random matrices. Bull. Am. Math. Soc. **46**, 377–396 (2009)
38. Tao, T., Vu, V.: Random matrices: Universality of local eigenvalue statistics up to the edge, preprint. http://arxiv.org/PS_cache/arxiv/pdf/0908/0908.1982v1.pdf
39. Wigner, E.: On the statistical distribution of the widths and spacings of nuclear resonance levels. Proc. Camb. Philol. Soc. **47**, 790–798 (1951)
40. Wigner, E.: Characteristic vectors of bordered matrices with infinite dimensions. Ann. Math. **2**(62), 548–564 (1955)
41. Wigner, E.: Statistical properties of real symmetric matrices. In: Canadian Mathematical Congress Proceedings, pp. 174–184. University of Toronto Press, Toronto (1957)
42. Wigner, E.: Characteristic vectors of bordered matrices with infinite dimensions. II. Ann. Math. Ser. 2 **65**, 203–207 (1957)
43. Wigner, E.: On the distribution of the roots of certain symmetric matrices. Ann. Math. Ser. 2 **67**, 325–327 (1958)
44. Wishart, J.: The generalized product moment distribution in samples from a normal multivariate population. Biometrika A **20**, 32–52 (1928)
45. Xiong, W.: The limiting spectral measure for the ensemble of generalized real symmetric period m-circulant matrices, senior thesis (advisor S.J. Miller), Williams College (2011)
46. Zvonkin, A.: Matrix integrals and map enumeration: An accessible introduction. Math. Comput. Model. **26**(8–10), 281–304 (1997)