Generating weights of modules of vector-valued modular forms for the Weil representation

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Part One: Vector-Valued Modular Forms

Vector-valued Modular Forms, Classically

Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}^n)$ denote a complex, finite-dimensional representation of the metaplectic group, and let $\mathfrak{H} = \{\tau \in \mathbb{Z} : \mathrm{Im}(\tau) > 0\}$ be the upper half-plane.

Definition

A weakly holomorphic ρ-valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}^n$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \phi^{2k} \rho(M) f(\tau)$$

for all $M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \in \mathrm{Mp}_2(\mathbb{Z})$. (Here, $\phi^2 = c\tau + d$.)

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We say that f is holomorphic if, in addition, it has a q-expansion at the cusp ∞.

Setup

- Geometric perspective due in part to Luca Candelori and Cameron Franc (earlier paper on $\mathrm{SL}_2(\mathbb{Z})$), and in part to Terry Gannon.
- Consider orbifold quotient $\mathrm{Mp}_2(\mathbb{Z})\backslash\mathfrak{H}$.
- Idea: Records stabilizer at each point: Generically $\mathbb{Z}/4\mathbb{Z}$; $\mathbb{Z}/8\mathbb{Z}$ at i; $\mathbb{Z}/12\mathbb{Z}$ at ω.

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Definition

*A **vector bundle** on $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$ consists of a vector bundle $\pi : \mathcal{V} \to \mathfrak{H}$ endowed with an action of $\mathrm{Mp}_2(\mathbb{Z})$ that*

- *is linear on fibers, and*
- *lifts the fractional linear transformation action on $\mathfrak{H}$.*

Geometric Interpretation of VVMFs

Definition

If $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$ is a representation and $k \in \frac{1}{2}\mathbb{Z}$, then define $\mathcal{V}_k(\rho) = \mathbb{C}^n \times \mathfrak{H}$, endowed with the action

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \cdot (v, \tau) := \left(\phi^{2k}\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) v, \frac{a\tau + b}{c\tau + d}\right).$$

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- Global sections of $\mathcal{V}_k(\rho)$ are weak holomorphic vector-valued modular forms.
- To get rid of “weak,” we need to put the cusp into the orbifold.

Geometric Interpretation of VVMFs

- Glue to $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{H}$ a coordinate chart at ∞: $C_4\backslash\!\backslash D$ with C_4 acting trivially on the unit disk D.
- Change of coordinates is $q = e^{2\pi i\tau}$.
- Obtain compactified metaplectic orbifold $\overline{\mathcal{M}}_{1/2}$.

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Definition

A vector bundle on $\overline{\mathcal{M}}_{1/2}$ consists of vector bundles on the two charts whose pullbacks agree on the overlap.

Choice of Exponents

- Recall: We want to give a vector bundle definition of VVMFs holomorphic at the cusp.
- So we want to extend $\mathcal{V}_k(\rho)$ to $\overline{\mathcal{M}}_{1/2}$.
- There multiple non-isomorphic ways to do this, and they depend on a choice of exponents.

Definition

Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$ and let I be a half-open interval of length 1. A choice of exponents for ρ and I is a matrix L such that $\rho(T) = \exp(2\pi i L)$, and so the eigenvalues of L all lie in I.

$$S = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i \right) \qquad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

Choice of Exponents

- We define an extension $\overline{\mathcal{V}}_{k,L}(\rho)$, which turns out only to depend on I.
- When $I = [0,1)$, global sections of $\overline{\mathcal{V}}_{k,L}(\rho)$ are vector-valued modular forms holomorphic at the cusp.
- From now on, we fix L with $I = [0,1)$ (a standard choice of exponents) and set $\overline{\mathcal{V}}_k(\rho) := \overline{\mathcal{V}}_{k,L}(\rho)$.

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Generating Weights

Denote by $M_k(\rho) = H^0(\overline{\mathcal{M}}_{1/2}, \overline{\mathcal{V}}_k)$ the vector space of holomorphic VVMFs of weight k for ρ. Define

$$M(\rho) = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} M_k(\rho). \tag{1}$$

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Theorem (Candelori, Franc, K)

(i) *$M(\rho)$ is a free module of rank $\dim \rho$ over $M(1)$.*

(ii) *If $k_1 \leq \ldots \leq k_n$, $k_j \in \frac{1}{2}\mathbb{Z}$, are the weights of the free generators, then*

$$\sum_j k_j = 12\,\mathrm{Tr}(L).$$

(iii) *If ρ is unitarizable, then $0 \leq k_j \leq 23/2$.*

Weil Representation

Finite Quadratic Modules

The approach we present to the Weil representation is due to Nils-Peter Skoruppa.

Definition

A finite quadratic module is a pair (D, q) consisting of a finite abelian group D together with a quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, whose associated bilinear form we denote by $b(x, y) := q(x + y) - q(x) - q(y)$.

We will compute the generating weights of the Weil representation associated to the finite quadratic module $A_{2p^r} := \left(\mathbb{Z}/2p^r\mathbb{Z}, \frac{x^2}{4p^r}\right)$.

Weil Representation

For (D, q) a finite quadratic module, let $\mathbb{C}[D]$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x \in D}$ of delta functions, where $\delta_x(y) = \delta_{xy}$. The Weil Representation

$$\rho_D : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}[D])$$

is defined with respect to this basis by

$$\rho_D(T)(\delta_x) = \exp(-2\pi i q(x))\delta_x,$$

$$\rho_D(S)(\delta_x) = \frac{\sqrt{i}^{\mathrm{sig(D)}}}{\sqrt{|D|}} \sum_{y \in D} \exp(2\pi i b(x, y))\delta_y.$$

Outline of Computation of Generating Weights for Weil Repn

- The metaplectic orbifold $\overline{\mathcal{M}}_{1/2}$ has the structure of a Deligne-Mumford stack. Riemann-Roch for Deligne-Mumford stacks gives us a complicated formula for the Euler characteristic $\dim H^0(\overline{\mathcal{M}}_{1/2}, \overline{\mathcal{V}}_k) - \dim H^1(\overline{\mathcal{M}}_{1/2}, \overline{\mathcal{V}}_k)$.

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- $\dim H^1(\overline{\mathcal{M}}_{1/2}, \overline{\mathcal{V}}_k)$ can be computed for all weights in our case.
- Terms in Riemann-Roch formula are computed: Most interesting is Tr(L).
- We now know $\dim M_k(\rho) = \dim H^0(\overline{\mathcal{M}}_{1/2}, \overline{\mathcal{V}}_k)$ for every k, which together determine the generating weights.

Computation of Tr(L)

Tr(L)

For the Weil representation $\rho = \rho_{A_{2p^r}}$,

$$\rho(T) = \exp\left(2\pi i \begin{pmatrix} \frac{-1}{4p^r} & 0 & \cdots & 0 \\ 0 & \frac{-4}{4p^r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-(2p^r)^2}{4p^r} \end{pmatrix}\right).$$

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Therefore,

$$\mathrm{Tr}(L) = \sum_{x=1}^{2p^r} \left\{\frac{-x^2}{4p^r}\right\}.$$

Distribution of Quadratic Residues Modulo p

- In $\{1, 2, \ldots, p-1\}$, there are the same number of squares and non-squares $\pmod{p}$.
- In $\{1, 2, \ldots, \frac{p-1}{2}\}$, there are:
  - the same number of squares and non-squares, if $p \equiv 1 \pmod{4}$;
  - more squares than non-squares, if $p \equiv 3 \pmod{4}$. Difference is $\left(2 - \left(\frac{2}{p}\right)\right) h_{\mathbb{Q}(\sqrt{-p})}$.

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  - more squares than non-squares, if $p \equiv 3 \pmod 4$. Difference is $\left(2 - \left(\frac{2}{p}\right)\right) h_{\mathbb{Q}(\sqrt{-p})}$.
- Lesser known: If $p \equiv 1 \pmod 4$, there are more squares than non-squares in $\{1, 2, \ldots, \frac{p-1}{4}\}$. Difference is $\frac{1}{2} h_{\mathbb{Q}(\sqrt{-p})}$.
- Both "more squares than non-squares" rely on Dirichlet's class number formula for negative discriminants.

Computation of Tr(L)

For an example of the computation, we look at the case of A_{2p} where $p \equiv 1 \pmod 4$. By elementary manipulations and modular arithmetic, we show that

$$\begin{aligned}\mathrm{Tr}(L) &= \sum_{x=1}^{2p^r} \left\{ \frac{-x^2}{4p^r} \right\} \\ &= p - \left\{ \frac{p}{4} \right\} + \sum_{a=1}^{\frac{p-1}{4}} \left(\frac{a}{p} \right).\end{aligned}$$

Then, the lesser-known "more squares than non-squares" result tells us that $\displaystyle\sum_{a=1}^{\frac{p-1}{4}} \left(\frac{a}{p} \right) = \frac{1}{2} h_{\mathbb{Q}(\sqrt{-p})}$.

Results on the Weil Representation

Formula for Tr(L)

Theorem (Candelori, Franc, K)

Suppose $p > 3$ is prime, let $r \geq 1$, and let L denote a standard choice of exponents for the Weil representation associated to the quadratic module A_{2p^r}. Then,

$$\mathrm{Tr}(L) = p^r - \left\{\frac{p^r}{4}\right\} p^{\lfloor \frac{r}{2} \rfloor} + \frac{p^{\lfloor \frac{r+1}{2} \rfloor} - 1}{2(p-1)} \cdot \begin{cases} h_p & \text{if } p \equiv 1 \mod 4 \\ 4h_p + 1 & \text{if } p \equiv 3 \mod 8 \\ 2h_p + 1 & \text{if } p \equiv 7 \mod 8 \end{cases} .$$

Formula for $\mathrm{Tr}(L)$

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Note that $\mathrm{Tr}(L) = p^r + O\left(p^{(r+1)/2}\right)$ $p \to \infty$ or as $r \to \infty$.

Table of Weight Multiplicities

Theorem (Candelori, Franc, K)

The generating weights for the Weil representation associated to a cyclic quadratic module of order $2p^r$ *are as shown in the table on the next slide, where*

$$\delta := \frac{1}{8}\left(2 + \left(\frac{-1}{p^r}\right)\right), \quad \epsilon_{\pm} := \frac{1}{6}\left(1 \pm \left(\frac{p^r}{3}\right)\right). \tag{2}$$

Table of Weight Multiplicities

Weights k	Multiplicities m_k
$1/2$	0
$3/2$	$\frac{13}{24}(p^r+1) - \frac{1}{2}\operatorname{Tr}(L) - \delta - \epsilon_+$
$5/2$	$\frac{15}{24}(p^r-1) - \frac{1}{2}\operatorname{Tr}(L) + \delta$
$7/2$	$\frac{17}{24}(p^r+1) - \frac{1}{2}\operatorname{Tr}(L) - \delta + \epsilon_+$
$9/2$	$\frac{19}{24}(p^r-1) - \frac{1}{2}\operatorname{Tr}(L) + \delta + \epsilon_-$
$11/2$	$\frac{1}{3}(p^r+1) + \epsilon_+$
$13/2$	$\frac{1}{3}(p^r-1) - \epsilon_-$
$15/2$	$-\frac{5}{24}(p^r+1) + \frac{1}{2}\operatorname{Tr}(L) + \delta - \epsilon_+$
$17/2$	$-\frac{7}{24}(p^r-1) + \frac{1}{2}\operatorname{Tr}(L) - \delta - \epsilon_-$
$19/2$	$-\frac{9}{24}(p^r+1) + \frac{1}{2}\operatorname{Tr}(L) + \delta$
$21/2$	$-\frac{11}{24}(p^r-1) + \frac{1}{2}\operatorname{Tr}(L) - \delta + \epsilon_-$
$23/2$	0

Limiting Distribution

Corollary (Candelori, Franc, K)

Let $\rho = \rho_{A_{2p^r}}$, where $p > 3$ is a prime, as above. Let m_k denote the multiplicity of the generating weight $k \in \frac{1}{2}\mathbb{Z}$ for $M(\rho)$. Then the values of $\lim_{p\to\infty} \frac{m_k}{\dim \rho}$ are as follows:

Weight k	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$	$\frac{21}{2}$	$\frac{23}{2}$
$\lim\limits_{p\to\infty} \frac{m_k}{\dim \rho}$	0	$\frac{1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$	$\frac{8}{48}$	$\frac{8}{48}$	$\frac{7}{48}$	$\frac{5}{48}$	$\frac{3}{48}$	$\frac{1}{48}$	0

Thank you!

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Thank you to my coauthors Cameron Franc and Luca Candelori, and to my Ph.D. advisor, Jeff Lagarias.

Candelori, L., Franc, C., and Kopp, G. Generating weights for the Weil representation attached to an even order cyclic quadratic module. Submitted, 2016. arxiv:1606.07844.

Generalization?

- Why not all orders or all quadratic modules?
- Computation of $\mathrm{Tr}(L)$ gets more complicated but is doable...
- But h^1 cannot be computed or related to h^0 in weight 1 by our methods. Our computation relies on a parity argument to ensure there are no forms of integral weight.
- This is the case whenever the signature $\mathrm{sig}(D)$ is even.