## Axioms for the Integers

The following list of axioms uniquely specify the integers $\mathbb{Z}$ with addition, multiplication, and ordering. (There are more minimalistic lists of axioms, the most famous of which is the Peano axioms for the natural numbers $\mathbb{N}$: `https://en.wikipedia.org/wiki/Peano_axioms`.) The integers are a set $\mathbb{Z}$ together with two binary operations $+$ and $\cdot$ (also written as concatenation), as well as a binary relation $<$, satisfying the following axioms.

(1) For all $a, b, c, \in \mathbb{Z}$, $a + (b + c) = (a + b) + c$.
(2) There exists $0 \in \mathbb{Z}$ such that for all $a \in \mathbb{Z}$, $a + 0 = a$.
(3) For each $a \in \mathbb{Z}$, there exists $-a \in \mathbb{Z}$ such that $a + (-a) = 0$.
(4) For all $a, b \in \mathbb{Z}$, $a + b = b + a$.
(5) For all $a, b, c \in \mathbb{Z}$, $a(bc) = (ab)c$.
(6) For all $a, b, c \in \mathbb{Z}$, $a(b + c) = ab + ac$.
(7) There exists $1 \in \mathbb{Z}$ such that for all $a \in \mathbb{Z}$, $a \cdot 1 = a$.
(8) For all $a, b \in \mathbb{Z}$, $ab = ba$.
(9) For all $a \in \mathbb{Z}$, $\text{not}(a < a)$.
(10) For all $a, b, c \in \mathbb{Z}$, if $a < b$ and $b < c$, then $a < c$.
(11) For all $a, b \in \mathbb{Z}$, one and only one of the following are true: $a < b$, $a = b$, or $b < a$.
(12) For all $a, b, c \in \mathbb{Z}$, if $a < b$, then $a + c < b + c$.
(13) $0 < 1$.
(14) [Induction] Let $\mathcal{S}(n)$ be a statementabout integers and suppose $\mathcal{S}(n_0)$ is true for some $n_0 \in \mathbb{Z}$. If for all $k \in \mathbb{Z}$ with $n_0 \leq k$, $\mathcal{S}(k)$ implies $\mathcal{S}(k + 1)$, then $\mathcal{S}(n)$ is true for all $n \in \mathbb{Z}$ with $n_0 \leq n$. (Here, $a \leq b$ is defined to mean $a < b$ or $a = b$.)

Certain subsets of these axioms will be used later in the course to define **groups** and **rings**, and reasoning formally with axioms will be important when proving things about these abstract objects.

As you may notice, the axioms contain certain undefined notions such as "=", "statement", and words denoting quantifiers (such as "for all") and logical connectors (such as "and"). In this course, we accept these notions as part of our logical foundation for doing mathematics. We will also treat set theory informally and use concepts like "set" and "subset" without defining them.

You will not need to proceed directly from these axioms to prove statements about integers on the problem sets. On the problem sets, you may assume basic arithmetic facts like (E1)–(E6) below.

You may try the following (optional) exercises to get practice working formally with axioms and additional practice with proofs by induction. Define the natural numbers $\mathbb{N} = \{n \in \mathbb{Z} : 0 < n\}$.

(E1) Prove that $-(-a) = a$ for all $a \in \mathbb{Z}$.
(E2) Prove that $a \cdot 0 = 0$ for all $a \in \mathbb{Z}$.
(E3) Prove that for all $a, b \in \mathbb{Z}$ and $c \in \mathbb{N}$, if $a < b$, then $ac < bc$.
(E4) Prove the following statement:

(NIBZO) There is no integer between $0$ and $1$.

(E5) Prove the well-ordering principal:

(WO) Every nonempty subset of $\mathbb{N}$ has a least element.

(E6) Define exponentiation, and prove that $a^{m+n} = a^m a^n$, $(ab)^n = a^n b^n$, and $(a^m)^n = a^{mn}$.
(E7) Show that axioms (1)–(13) together with (WO) imply (14).
(E8$^*$) Show that axioms (1)–(13) together with (NIBZO) do not imply (14). Do so by constructing a number system other than $\mathbb{Z}$ that satisfies (1)–(13) and (NIBZO), but not (14).