Wannabe modularity, the Shintani-Faddeev cocycle, and Stark units

Gene S. Kopp

Purdue University

Southern Regional Number Theory Conference
Louisiana State University
March 13, 2022

Special Thanks to NSA and NSF

The q-Pochhammer symbol

The (infinite) q-Pochhammer symbol is

$$(w, q)_\infty = \prod_{k=0}^{\infty} (1 - wq^k).$$

Write $w = \mathrm{e}(z) = e^{2\pi i z}$ and $q = \mathrm{e}(\tau) = e^{2\pi i \tau}$, and set

$$\varpi(z, \tau) = (w, q)_\infty.$$

Question

How does $\varpi(z, \tau)$ transform under:

- elliptic transformations $z \mapsto z + m\tau + n$?
- modular transformations $(z, \tau) \mapsto \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$?

The Jacobi triple product and theta functions

The Jacobi triple product says that

$$\prod_{k=1}^{\infty}(1-q^k)(1-wq^{k-\frac{1}{2}})(1-w^{-1}q^{k-\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} w^n q^{\frac{n^2}{2}}.$$

After a change of variables, the Jacobi triple product is equivalent to the formula

$$\varpi(z,\tau)\varpi(-z,\tau) = -i\mathrm{e}\left(-\frac{\tau}{12}\right)\left(\mathrm{e}\left(\frac{z}{2}\right)-\mathrm{e}\left(-\frac{z}{2}\right)\right)\frac{\vartheta_1(z,\tau)}{\eta(\tau)}, \text{ where}$$

$\eta(\tau) = \mathrm{e}\left(\frac{\tau}{24}\right)\prod_{k=1}^{\infty}(1-\mathrm{e}(k\tau))$ is the Dedekind eta function, and

$$\vartheta_1(z,\tau) = -\sum_{n=-\infty}^{\infty}\mathrm{e}\left(\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right)$$

is a Jacobi theta function.

The Jacobi triple product and theta functions

$$\varpi(z,\tau)\varpi(-z,\tau) = -i\,\mathrm{e}\left(-\frac{\tau}{12}\right)\left(\mathrm{e}\left(\frac{z}{2}\right) - \mathrm{e}\left(-\frac{z}{2}\right)\right) f(z,\tau),$$

$$\text{where } f(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta(\tau)}.$$

- The function $f(z,\tau)$ is a meromorphic Jacobi form of weight 0 with character—it transforms “nicely” under elliptic and modular transformations.
- $\varpi(z,\tau)$ is essentially “half” of a meromorphic Jacobi form—does it have interesting transformation properties on its own?

Elliptic transformations of $\varpi(z,\tau)$

$f(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta(\tau)}$; $\varpi(z,\tau) = (\mathrm{e}(z)\,,\mathrm{e}(\tau))_\infty$.

$$f(z+1,\tau) = -f(z,\tau) \qquad \varpi(z+1,\tau) = \varpi(z,\tau)$$

$$f(z+\tau,\tau) = -\mathrm{e}\left(-\frac{\tau}{2} - z\right) f(z,\tau) \qquad \varpi(z+\tau,\tau) = (1-\mathrm{e}(z))^{-1}\varpi(z,\tau)$$

General elliptic transformation: For any $m, n \in \mathbb{Z}$,

$$f(z+m\tau+n,\tau) = (-1)^{m+n}\mathrm{e}\left(-\frac{m^2}{2}\tau - z\right) f(z,\tau)$$

$$\varpi(z+m\tau+n,\tau) = (\mathrm{e}(z)\,,\mathrm{e}(\tau))_m^{-1}\,\varpi(z,\tau)$$

Modular transformations of $\varpi(z,\tau)$

$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T\rangle$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

$S \cdot (z,\tau) = \left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$ and $T \cdot (z,\tau) = (z, \tau+1)$.

Transformations under T:

$$f(z,\tau+1) = \mathrm{e}\left(\frac{1}{8}\right) f(z,\tau) \qquad \varpi(z,\tau+1) = \varpi(z,\tau)$$

Transformations under S:

$$f\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i\,\mathrm{e}\left(\frac{z^2}{2\tau}\right) f(z,\tau)$$
$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (???)\varpi(z,\tau)$$

Modular transformations of $\varpi(z,\tau)$, continued

Theorem (Shintani, 1977 (rephrased))

The modular transformation law for q-Pochhammer symbol under the S-matrix is

$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \mathrm{e}\left(\frac{\tau - 3 + \tau^{-1}}{24} + \frac{(\tau - z)(1 - z)}{4\tau}\right)\left(1 - \mathrm{e}\left(\frac{z}{\tau}\right)\right) \cdot \mathrm{Sin}_2(z,\tau)\varpi(z,\tau),$$

where $\mathrm{Sin}_2(z,\tau)$ is the double sine function, a meromorphic function of $z \in \mathbb{C}$ and $\tau \in \mathbb{C} \setminus (-\infty, 0]$.

- Shintani called this result a “Proposition”.
- The term “double sine function” is due to Kurokawa. (Shintani called the function F.)

Double sine function

Shintani's definition of $\mathrm{Sin}_2(z,\tau)$ is in terms of functions previously defined by Barnes. The **double zeta function** is

$$\zeta_2(s,z;\omega_1,\omega_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(z+\omega_1 m+\omega_2 n)^{-s}.$$

The **double gamma function** is

$$\Gamma_2(z;\omega_1,\omega_2) = \rho_2(\omega_1,\omega_2)\exp\left(\frac{d}{ds}\zeta_2(s,z;\omega_1,\omega_2)\bigg|_{s=0}\right).$$

The **double sine function** is

$$\mathrm{Sin}_2(z;\omega_1,\omega_2) = \frac{\Gamma_2(\omega_1+\omega_2-z;\omega_1,\omega_2)}{\Gamma_2(z;\omega_1,\omega_2)}.$$

Because $\mathrm{Sin}_2(\alpha z;\alpha\omega_1,\alpha\omega_2) = \mathrm{Sin}_2(z;\omega_1,\omega_2)$, we lose no generality by defining $\mathrm{Sin}_2(z,\tau) = \mathrm{Sin}_2(z;\tau,1)$.

History—number theory side

- Shintani (1977) used his double sine function to give “Kronecker limit formulas” for first derivatives at zero of partial zeta functions attached to real quadratic number fields, as products of double sine values.
- Shintani also generalized to totally real fields.
- Kurowawa and Koyama studied these functions further and found other number-theoretic applications.
- Zagier, Hayes, Sczech, Tangedal, Yamamoto, and others have refined or reinterpreted Shintani’s formulas.

History—physics side

- Meanwhile, Faddeev (1994) independently discovered the double sine function and its relation to the q-Pochhammer symbol. He called it the noncompact quantum dilogarithm and used it in conformal field theory.
- The physics literature includes many fascinating and highly nontrivial integral formulas involving the noncompact quantum dilogarithm.
- Recently, Sarkissian and Spiridonov (2020) considered a general modular quantum dilogarithm.

General modular transformations of $\varpi(z,\tau)$

Theorem (K 2022+, Sarkissian-Spiridonov 2020)

For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, there is a meromorphic function $\sigma_\gamma(z,\tau)$ on $\mathbb{C} \times (\mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\})$ with the property that

$$\varpi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \sigma_\gamma(z,\tau)\varpi(z,\tau).$$

Definition (K 2022+)

We call the map $\gamma \mapsto \sigma_\gamma(z,\tau)$ the Shintani-Faddeev Jacobi cocycle.

So...

Questions:

- In what sense is $\gamma \mapsto \sigma_\gamma(z, \tau)$ a cocycle?
- In what sense is $\varpi(z, \tau)$ modular or Jacobi-form-like?

I'm still trying to find the best answers to these questions—but I will present my current working formalism.

Generalized group cohomology

Suppose Γ is a group and A is a $\mathbb{Z}[\Gamma]$-module. The first cohomology group is

$$H^1(\Gamma, A) = \frac{\{a : \Gamma \to A \mid a_{\gamma_1\gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\{a : \Gamma \to A \mid a_\gamma = c^\gamma c^{-1} \text{ for some } c \in A\}}.$$

Definition (K 2022+)

If we also have a function $B : \Gamma \to \{\text{subgroups of } A\}$, we may define a generalized first cohomology group

$$H^1_B(\Gamma, A) = \frac{\{a : \Gamma \to A \mid a_\gamma \in B_\gamma, a_{\gamma_1\gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\left\{a : \Gamma \to A \mid a_\gamma = c^\gamma c^{-1} \text{ for some } c \in \bigcap_{\gamma\in\Gamma} B_\gamma\right\}}.$$

Weight cocycles

Now let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Definition (K 2022+)

A system of domains is a collection of connected open sets $\mathbb{H} \subseteq D_\gamma \subseteq \mathbb{C} \cup \{\infty\}$ indexed by $\gamma \in \Gamma$. A weight cocycle (or “generalized factor of automorphy”) is a collection of nonzero meromorphic functions $w_\gamma : D_\gamma \to \mathbb{C}$ such that

$$w_{\gamma_1\gamma_2}(\tau) = w_{\gamma_1}(\gamma_2 \cdot \tau) w_{\gamma_2}(\tau).$$

In other words, w_γ defines a cohomology class

$$[w_\gamma] \in H^1_{\mathcal{B}}(\Gamma, \mathcal{M}_{\mathbb{H}}^\times)$$

where $\mathcal{M}_D$ denotes the ring of meromorphic functions on D, and $\mathcal{B}_\gamma = \mathcal{M}_{D_\gamma}^\times$.

Wannabe modular forms

Wannabe modular forms

Definition (K 2022+)

Let $w_\gamma(\tau)$ be a weight cocycle for some system of domains D_γ. A meromorphic complex-valued function $f : \mathbb{H} \to \mathbb{C}$ is wannabe modular of weight $w_\gamma(\tau)$ if

$$f(\gamma \cdot \tau) = w_\gamma(\tau) f(\tau)$$

for all $\tau \in \mathbb{H}$.

First example: For every $k \in \mathbb{Z}$, the function $j_\gamma(\tau)^k = (c\tau + d)^k$ is a weight cocycle for the constant system of domains $D_\gamma = \mathbb{C} \cup \{\infty\}$, and any modular form of weight k is a wannabe modular form of weight $j_\gamma(\tau)^k$.

The Shintani-Faddeev modular cocycle

Second example: For $\mathbf{p} = (p_1, p_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$, let

$$\varpi_{\mathbf{p}}(\tau) = \varpi(p_1\tau + p_2, \tau) = (\mathrm{e}(p_1\tau + p_2), \mathrm{e}(\tau))_\infty.$$

Theorem (K 2022+)

The function $\varpi_{\mathbf{p}}(\tau)$ is a wannabe modular form for a congruence subgroup of level N. The associated weight cocycle $\gamma \mapsto \mathfrak{w}^{\mathbf{p}}_\gamma(\tau)$, called the Shintani-Faddeev modular cocycle (with characteristics $\mathbf{p}$), is valued in meromorphic functions $\mathfrak{w}^{\mathbf{p}}_\gamma(\tau)$ on $\widetilde{D}_\gamma$, where

$$\widetilde{D}_\gamma = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

Wannabe Jacobi forms

We may define wannabe Jacobi forms in a similar manner (details omitted).

Theorem (rephrasing of transformation laws)

The function $\varpi(z,\tau)$ is a wannabe Jacobi form with weight cocycle $\gamma \mapsto \sigma_\gamma(z,\tau)$ and system of domains

$$D_\gamma = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0 \text{ and } d > 0, \\ \mathbb{H} & \text{if } c = 0 \text{ and } d < 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

Hilbert’s 12 problem and the Stark conjectures

- Kronecker-Weber theorem says that the abelian Galois extensions of $\mathbb{Q}$ are generated by the values of $e(z) = e^{2\pi iz}$ at rational values of z.
- Hilbert’s 12th problem asks for an extension of Kronecker-Weber where $\mathbb{Q}$ is replaced by a general number field K.
- Given any K, Hilbert wanted analytic functions that play the role of $e(z)$.
- Harold Stark conjectured in a series of papers (1971–1980) that $\exp(cZ(1))$, for certain linear combinations $Z(s)$ of Hecke L-functions of K, generate abelian extensions of K.

L-functions at $s = 1$: rational example

The following formula can be proved using calculus. Try it!

Example

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. The right-hand side involves $\varepsilon = 1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.

L-functions at $s = 1$: imaginary quadratic example

The following formula is proved using the theory of complex multiplication for elliptic curves.

Example

$$\sum\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{e(m/5)-e(2m/5)}{m^2+mn+n^2} = \frac{2\pi}{\sqrt{3}}\log\left(\varepsilon^{1/5}\right)$$

where $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65+29\sqrt{5})}$.

The left-hand side is a linear combination of Hecke L-values at $s = 1$ for $\mathbb{Q}(\sqrt{-3})$. The right-hand side involves an algebraic unit ε in the ray class field modulo (5) for $\mathbb{Q}(\sqrt{-3})$.
This example is related to the 5-torsion points of the CM elliptic curve $y^2 = x^3 + 1$. This elliptic curve has complex multiplication by $\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$.

L-functions at $s = 1$: real quadratic example

The following formula is an open conjecture!

Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n\in\mathbb{Z} \\ -\frac{5}{3}m \le n < \frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where $\varepsilon \approx 3.890861714$ is a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

The number ε is an algebraic unit in the ray class field of $\mathbb{Q}(\sqrt{3})$ modulo $5\infty_2$. This conjecture is part of the Stark conjectures.

Ray class groups and ray class fields

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathfrak{m}$ be a ideal in $\mathcal{O}_K$, and let Σ be a subset of the real embeddings of K.

Definition (Ray class group modulo $\mathfrak{m}, \Sigma$)

$$\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K) = \frac{\{\text{fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{m}\}}{\{a\mathcal{O}_K \text{ s.t. } a \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}}$$

Class field theory associates to $\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$ a ray class field $H_{\mathfrak{m},\Sigma}$, an abelian extension of K with Galois group $\mathrm{Gal}(H_{\mathfrak{m},\Sigma}/K) = \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$. Varying $\mathfrak{m}$ and Σ, the ray class fields are cofinal among all abelian extensions of K.

Zeta functions associated to ray classes

Definition

For $\mathscr{A} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$, the associated zeta function is

$$\zeta(s, \mathscr{A}) = \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in \mathscr{A}}} \mathrm{Nm}(\mathfrak{a})^{-s}.$$

Let $R \in \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$ be the ideal class

$$\mathscr{R} = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}.$$

Definition

For $\mathscr{A} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$, the associated differenced zeta function is

$$Z_{\mathscr{A}}(s) = \zeta(s, \mathscr{A}) - \zeta(s, \mathscr{R}\mathscr{A}).$$

Rank 1 abelian Stark conjecture over a real quadratic field

Conjecture (Stark, 1976, special case)

Setup:

- Let K be a real quadratic number field.
- Consider $0 \neq \mathfrak{m} \leq \mathcal{O}_K$ with the property that, if $\varepsilon \in \mathcal{O}_K^\times$ and $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$, then one of ε or $-\varepsilon$ is totally positive.
- Let $\mathscr{A}$ be a ray ideal class in $\mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K)$.
- Let $H_{\mathfrak{m}\infty_j}$ be the ray class field of K modulo $\mathfrak{m}\infty_j$.
- Let ρ_j be the real embedding of $H_{\mathfrak{m}\infty_1\infty_2}$ associated to ∞_j.

Then,

(1) $Z'_{\mathscr{A}}(0) = \log(\rho_1(\varepsilon_{\mathscr{A}}))$ for a unit $\varepsilon_{\mathscr{A}} \in H_{\mathfrak{m}\infty_2}$.

(2) The units $\varepsilon_{\mathscr{A}}$ are compatible with the isomorphism $\mathrm{Art} : \mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K) \to \mathrm{Gal}(H_{\mathfrak{m}\infty_2}/K)$. Specifically, $\varepsilon_{\mathscr{A}} = \varepsilon_{\mathrm{id}}{}^{\mathrm{Art}(\mathscr{A})}$.

Hilbert's 12th problem and cocycles

- Hilbert's 12th problem was solved for imaginary quadratic fields using complex multiplication values of modular functions (0-cocycles for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$).
- **Hope:** Use "real multiplication values" of 1-cocycles to solve Hilbert's 12th problem for real quadratic fields.
- In fact, it's no longer just a hope: Darmon, Pozzi, and Vonk (2021) prove algebraicity results for real multiplication values of their *p*-adic Dedekind-Rademacher cocycle using the groundbreaking work of Dasgupta and Kakde (2021) on the Gross-Stark and Brumer-Stark conjectures.
- But what about the complex (not *p*-adic) setting?

Real multiplication values of a weight cocycle

- Let $w_\gamma(\tau)$ be a weight cocycle.
- Consider a real quadratic number β.
- Suppose $\gamma \in \Gamma$ is the "positive" generator for the stabilizer of β in Γ .
- Then, the values $w_\gamma(\beta)$ is the real multiplication value of w at β.

Real multiplication values of $\varpi_\gamma^{\mathbf{p}}(\tau)$

Theorem (K 2022+)

Let K be a real quadratic field, $N \in \mathbb{N}$. Let $\mathscr{A} \in \mathrm{Cl}(\mathcal{O}_K)$, choose some $\mathfrak{b} \in \mathscr{A}^{-1}$ coprime to $N\mathcal{O}_K$, and write $\mathfrak{b} = \mathbb{Z} + \beta\mathbb{Z}$ for some $\beta \in K$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the positive generator for $\mathrm{stab}(\beta)$ in $\Gamma(N)$. Let $\beta_0 \in \mathcal{O}_K$ such that $\beta \equiv \beta_0 \pmod N$. For $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, let

$$\mathscr{A}_{\mathbf{p}} = \{\alpha\mathcal{O}_K : \alpha \equiv p_1\beta + p_2 \pmod N \text{ and } \rho_2(\alpha) > 0\} \in \widetilde{\mathrm{Cl}}_{N\infty_2}(\mathcal{O}_K).$$

For an easily computable integer n ($= 1$ if $\beta_0\mathcal{O}_K + N\mathcal{O}_K = \mathcal{O}_K$),

$$n \exp(Z'_{N\infty_2}(0, \mathfrak{b}^{-1}\mathscr{A}_{\mathbf{p}})) = \left(\zeta_8^{-1}\epsilon(\gamma)\mathrm{e}\left(\frac{N+1}{4}\left(bp_1^2 - (a-d)p_1p_2 - cp_2^2\right)\right)\varpi_\gamma^{N^{-1}\mathbf{p}}(\beta)\right)^2.$$

Proof outline

- Use Jacobi triple product and modular properties of $\vartheta_1(z,\tau)$ and $\eta(\tau)$ to establish a relation between $\varpi_\gamma^{\mathbf{p}}(\tau)$ and $\varpi_\gamma^{-\mathbf{p}}(\tau)$.
- Write Tangedal's version of Shintani's formula (involving the Hirzebruch-Jung continued fraction of β) in terms of the Shintani-Faddeev cocycle, and use the cocycle condition to "telescope" the product.
- One is left with a complicated-looking root of unity factor...
- ...that may be simplified greatly using the combinatorics of continued fraction expansions and the relation between $\varpi_\gamma^{\mathbf{p}}(\beta)$ and $\varpi_\gamma^{-\mathbf{p}}(\beta)$ previously established.
- The maximal order $\mathcal{O}_K$ may also be replaced by an arbitrary order $\mathcal{O}$.

Future directions

- Big picture: Want to prove an algebraicity result for $\mathfrak{w}^{\mathbf{p}}_{\gamma}(\beta)$.
- Work out (conjecturally, at least) the action of the Artin map on the sign under the square root.
- Understand cohomology group containing $\gamma \mapsto \mathfrak{w}^{\mathbf{p}}_{\gamma}(\tau)$, that is, $H^1_{\widetilde{\mathcal{B}}}(\Gamma(N), \mathcal{M}^{\times}_{\mathbb{H}})$ with $\widetilde{\mathcal{B}}_{\gamma} = \mathcal{M}^{\times}_{\widetilde{D}_{\gamma}}$ (or a smaller group, if this turns out to be too big).
- Connect complex and p-adic cocycles.
- Try to do something with formulas and perspectives originating in conformal field theory literature. Connect CFT to CFT?

Thank you!

Thank you for listening! Any questions?