Workshop 7: Field Extensions

Friday, April 15, 2022

Discuss the following problems with your neighbors and solve them. You might not get to all the problems in class. Some may reappear on homework.

CORE EXERCISES

(1) Let $K = \mathbb{Q}[\sqrt{2}]$.
  (a) Show that K is a field.
  (b) Explain why $\mathcal{B} = \{1, \sqrt{2}\}$ is a basis for K as a vector space over $\mathbb{Q}$.
  (c) Let $\alpha \in K$. Show that the function $T_\alpha : K \to K$ defined by $T_\alpha(\zeta) = \alpha\zeta$ is a $\mathbb{Q}$-linear transformation from K to K.
  (d) Suppose $\alpha = a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$. Find the matrix M_α representing the linear transformation T_α in the basis $\mathcal{B}$.
  (e) Find the eigenvalues of M_α. What do you notice?
  (f) Prove that $K \cong \mathbb{Q}[x]/(x^2 - 2)$.

(2) Let $K = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$.
  (a) Show that L is a field.
  (b) Explain why $\mathcal{B} = \{1, \sqrt{3}\}$ is a basis for L as a vector space over K, whereas $\widetilde{\mathcal{B}} = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis for L as a vector space over $\mathbb{Q}$.
  (c) Let $\alpha \in L$. Show that the function $T_\alpha : L \to L$ defined by $T_\alpha(\zeta) = \alpha\zeta$ is a K-linear transformation from L to L and is also a $\mathbb{Q}$-linear transformation from L to L.
  (d) Suppose $\alpha = a + b\sqrt{3}$ for $a, b \in K$. Find the matrix M_α representing the K-linear transformation T_α in the basis $\mathcal{B}$.
  (e) Suppose $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ for $a, b, c, d \in \mathbb{Q}$. Find the matrix M_α representing the $\mathbb{Q}$-linear transformation T_α in the basis $\widetilde{\mathcal{B}}$.
  (f) Find some $\gamma \in L$ such that $L = \mathbb{Q}[\gamma]$.
  (g) Find a polynomial $g(x) \in \mathbb{Q}[x]$ such that $L \cong \mathbb{Q}[x]/(g(x))$.

FURTHER EXERCISES

(3) Let $g(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree d, and let $F = \mathbb{Q}[x]/(g(x))$.
  (a) Show that F is a field.
  (b) Explain why $\mathcal{B} = \{1, x, \ldots, x^{d-1}\}$ is a basis for F as a vector space over $\mathbb{Q}$.
  (c) Let $a(x) \in F$. Show that the function $T_{a(x)} : F \to F$ defined by $T_{a(x)}(f(x)) = a(x)f(x)$ defines a $\mathbb{Q}$-linear transformation from F to F.
  (d) Find the matrix M_x representing the F-linear transformation T_x in the basis $\mathcal{B}$.
  (e) Suppose $a(x) = \sum_{j=0}^{d-1} a_j x^j$ for $a_j \in \mathbb{Q}$. Let $M_{a(x)}$ be the matrix representing the F-linear transformation $T_{a(x)}$ in the basis $\mathcal{B}$. Write $M_{a(x)}$ in terms of M_x.

(4*) For any field F, the field of rational functions in one variable with coefficeints in F is denoted by

$$F(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in F[x] \text{ and } g(x) \neq 0 \right\}.$$

  (a) Find a basis for $\mathbb{C}(x)$ as a vector space over $\mathbb{C}$.
  (b) Find a basis for $\mathbb{R}(x)$ as a vector space over $\mathbb{R}$.
  (c) Find a basis for $\mathbb{Q}(x)$ as a vector space over $\mathbb{Q}$.