

Wannabe modularity, the Shintani-Faddeev cocycle, and Stark units

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The q -Pochhammer symbol

The (infinite) q -Pochhammer symbol is

$$(w, q)_\infty = \prod_{k=0}^{\infty} (1 - wq^k).$$

Write $w = e(z) = e^{2\pi iz}$ and $q = e(\tau) = e^{2\pi i\tau}$, and set

$$\varpi(z, \tau) = (w, q)_\infty.$$

Question

How does $\varpi(z, \tau)$ transform under:

- elliptic transformations $z \mapsto z + m\tau + n$?
- modular transformations $(z, \tau) \mapsto \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)$?

The Jacobi triple product and theta functions

The Jacobi triple product says that

$$\prod_{k=1}^{\infty} (1 - q^k)(1 - wq^{k-\frac{1}{2}})(1 - w^{-1}q^{k-\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} w^n q^{\frac{n^2}{2}}.$$

After a change of variables, the Jacobi triple product is equivalent to the formula

$$\varpi(z, \tau)\varpi(-z, \tau) = -ie\left(-\frac{\tau}{12}\right) \left(e\left(\frac{z}{2}\right) - e\left(-\frac{z}{2}\right)\right) \frac{\vartheta_1(z, \tau)}{\eta(\tau)}, \text{ where}$$

$\eta(\tau) = e\left(\frac{\tau}{24}\right) \prod_{k=1}^{\infty} (1 - e(k\tau))$ is the Dedekind eta function, and

$$\vartheta_1(z, \tau) = - \sum_{n=-\infty}^{\infty} e\left(\frac{1}{2} \left(n + \frac{1}{2}\right)^2 \tau + \left(n + \frac{1}{2}\right) \left(z + \frac{1}{2}\right)\right)$$

is a Jacobi theta function.

The Jacobi triple product and theta functions

$$\varpi(z, \tau) \varpi(-z, \tau) = -i e\left(-\frac{\tau}{12}\right) \left(e\left(\frac{z}{2}\right) - e\left(-\frac{z}{2}\right)\right) f(z, \tau),$$

where $f(z, \tau) = \frac{\vartheta_1(z, \tau)}{\eta(\tau)}$.

- The function $f(z, \tau)$ is a meromorphic Jacobi form of weight 0 with character—it transforms “nicely” under elliptic and modular transformations.
- $\varpi(z, \tau)$ is essentially “half” of a meromorphic Jacobi form—does it have interesting transformation properties on its own?

Elliptic transformations of $\varpi(z, \tau)$

$$f(z, \tau) = \frac{\vartheta_1(z, \tau)}{\eta(\tau)}; \varpi(z, \tau) = (e(z), e(\tau))_\infty.$$

$$f(z+1, \tau) = -f(z, \tau) \quad \varpi(z+1, \tau) = \varpi(z, \tau)$$

$$f(z+\tau, \tau) = -e\left(-\frac{\tau}{2} - z\right) f(z, \tau) \quad \varpi(z+\tau, \tau) = (1 - e(z))^{-1} \varpi(z, \tau)$$

General elliptic transformation: For any $m, n \in \mathbb{Z}$,

$$f(z + m\tau + n, \tau) = (-1)^{m+n} e\left(-\frac{m^2}{2}\tau - z\right) f(z, \tau)$$

$$\varpi(z + m\tau + n, \tau) = (e(z), e(\tau))_m^{-1} \varpi(z, \tau)$$

Modular transformations of $\varpi(z, \tau)$

$\text{SL}_2(\mathbb{Z}) = \langle S, T \rangle$, where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$S \cdot (z, \tau) = \left(\frac{z}{\tau}, -\frac{1}{\tau} \right)$ and $T \cdot (z, \tau) = (z, \tau + 1)$.

Transformations under T :

$$f(z, \tau + 1) = e\left(\frac{1}{8}\right) f(z, \tau) \quad \varpi(z, \tau + 1) = \varpi(z, \tau)$$

Transformations under S :

$$f\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i e\left(\frac{z^2}{2\tau}\right) f(z, \tau)$$
$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (\text{??}) \varpi(z, \tau)$$

Modular transformations of $\varpi(z, \tau)$, continued

Theorem (Shintani, 1977 (rephrased))

The modular transformation law for q -Pochhammer symbol under the S -matrix is

$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = e\left(\frac{\tau - 3 + \tau^{-1}}{24} + \frac{(\tau - z)(1 - z)}{4\tau}\right) \left(1 - e\left(\frac{z}{\tau}\right)\right) \cdot \text{Sin}_2(z, \tau) \varpi(z, \tau),$$

where $\text{Sin}_2(z, \tau)$ is the **double sine function**, a meromorphic function of $z \in \mathbb{C}$ and $\tau \in \mathbb{C} \setminus (-\infty, 0]$.

- Shintani called this result a “Proposition”.
- The term “double sine function” is due to Kurokawa. (Shintani called the function F .)

Double sine function

Shintani's definition of $\text{Sin}_2(z, \tau)$ is in terms of functions previously defined by Barnes. The **double zeta function** is

$$\zeta_2(s, z; \omega_1, \omega_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (z + \omega_1 m + \omega_2 n)^{-s}.$$

The **double gamma function** is

$$\Gamma_2(z; \omega_1, \omega_2) = \rho_2(\omega_1, \omega_2) \exp\left(\left.\frac{d}{ds} \zeta_2(s, z; \omega_1, \omega_2)\right|_{s=0}\right).$$

The **double sine function** is

$$\text{Sin}_2(z; \omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - z; \omega_1, \omega_2)}{\Gamma_2(z; \omega_1, \omega_2)}.$$

Because $\text{Sin}_2(\alpha z; \alpha \omega_1, \alpha \omega_2) = \text{Sin}_2(z; \omega_1, \omega_2)$, we lose no generality by defining $\text{Sin}_2(z, \tau) = \text{Sin}_2(z; \tau, 1)$.

History—number theory side

- Shintani (1977) used his **double sine function** to give “Kronecker limit formulas” for first derivatives at zero of partial zeta functions attached to real quadratic number fields, as products of double sine values.
- Shintani also generalized to totally real fields.
- Kurowawa and Koyama studied these functions further and found other number-theoretic applications.
- Zagier, Hayes, Sczech, Tangedal, Yamamoto, and others have refined or reinterpreted Shintani’s formulas.

History—physics side

- Meanwhile, Faddeev (1994) independently discovered the double sine function and its relation to the q -Pochhammer symbol. He called it the **noncompact quantum dilogarithm** and used it in conformal field theory.
- The physics literature includes many fascinating and highly nontrivial integral formulas involving the noncompact quantum dilogarithm.
- Recently, Sarkissian and Spiridonov (2020) considered a **general modular quantum dilogarithm**.

General modular transformations of $\varpi(z, \tau)$

Theorem (K 2022+, Sarkissian-Spiridonov 2020)

For each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there is a meromorphic function $\sigma_\gamma(z, \tau)$ on $\mathbb{C} \times (\mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\})$ with the property that

$$\varpi\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \sigma_\gamma(z, \tau) \varpi(z, \tau).$$

Definition (K 2022+)

We call the map $\gamma \mapsto \sigma_\gamma(z, \tau)$ the **Shintani-Faddeev Jacobi cocycle**.

So...

Questions:

- In what sense is $\gamma \mapsto \sigma_\gamma(z, \tau)$ a cocycle?
- In what sense is $\varpi(z, \tau)$ modular or Jacobi-form-like?

I'm still trying to find the best answers to these questions—but I will present my current working formalism.

Generalized group cohomology

Suppose Γ is a group and A is a $\mathbb{Z}[\Gamma]$ -module. The first cohomology group is

$$H^1(\Gamma, A) = \frac{\{a : \Gamma \rightarrow A \mid a_{\gamma_1 \gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\{a : \Gamma \rightarrow A \mid a_{\gamma} = c^{\gamma} c^{-1} \text{ for some } c \in A\}}.$$

Definition (K 2022+)

If we also have a function $B : \Gamma \rightarrow \{\text{subgroups of } A\}$, we may define a **generalized first cohomology group**

$$H_B^1(\Gamma, A) = \frac{\{a : \Gamma \rightarrow A \mid a_{\gamma} \in B_{\gamma}, a_{\gamma_1 \gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\{a : \Gamma \rightarrow A \mid a_{\gamma} = c^{\gamma} c^{-1} \text{ for some } c \in \bigcap_{\gamma \in \Gamma} B_{\gamma}\}}.$$

Weight cocycles

Now let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Definition (K 2022+)

A **system of domains** is a collection of connected open sets $\mathbb{H} \subseteq D_\gamma \subseteq \mathbb{C} \cup \{\infty\}$ indexed by $\gamma \in \Gamma$. A **weight cocycle** (or “generalized factor of automorphy”) is a collection of nonzero meromorphic functions $w_\gamma : D_\gamma \rightarrow \mathbb{C}$ such that

$$w_{\gamma_1\gamma_2}(\tau) = w_{\gamma_1}(\gamma_2 \cdot \tau) w_{\gamma_2}(\tau).$$

In other words, w_γ defines a cohomology class

$$[w_\gamma] \in H^1_B(\Gamma, \mathcal{M}_{\mathbb{H}}^\times)$$

where \mathcal{M}_D denotes the ring of meromorphic functions on D , and $\mathcal{B}_\gamma = \mathcal{M}_{D_\gamma}^\times$.

The q -Pochhammer symbol
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Wannabe modularity
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Interlude: rational values
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Stark conjectures
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Real multiplication values
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Wannabe modular forms



Wannabe modular forms

Definition (K 2022+)

Let $w_\gamma(\tau)$ be a weight cocycle for some system of domains D_γ . A meromorphic complex-valued function $f : \mathbb{H} \rightarrow \mathbb{C}$ is **wannabe modular of weight $w_\gamma(\tau)$** if

$$f(\gamma \cdot \tau) = w_\gamma(\tau) f(\tau)$$

for all $\tau \in \mathbb{H}$.

First example: For every $k \in \mathbb{Z}$, the function $j_\gamma(\tau)^k = (c\tau + d)^k$ is a weight cocycle for the constant system of domains $D_\gamma = \mathbb{C} \cup \{\infty\}$, and any modular form of weight k is a wannabe modular form of weight $j_\gamma(\tau)^k$.

The Shintani-Faddeev modular cocycle

Second example: For $\mathbf{r} = (r_1, r_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$, let

$$\varpi_{\mathbf{r}}(\tau) = \varpi(r_1\tau + r_2, \tau) = (\mathbf{e}(r_1\tau + r_2), \mathbf{e}(\tau))_\infty.$$

Theorem (K 2022+)

The function $\varpi_r(\tau)$ is a wannabe modular form for a congruence subgroup of level N . The associated weight cocycle $\gamma \mapsto \varpi_\gamma^r(\tau)$, called the **Shintani-Faddeev modular cocycle (with characteristics r)**, is valued in meromorphic functions $\varpi_\gamma^r(\tau)$ on \widetilde{D}_γ , where

$$\widetilde{D}_\gamma = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

Wannabe Jacobi forms

We may define wannabe Jacobi forms in a similar manner (details omitted).

Theorem (rephrasing of transformation laws)

The function $\varpi(z, \tau)$ is a wannabe Jacobi form with weight cocycle $\gamma \mapsto \sigma_\gamma(z, \tau)$ and system of domains

$$D_\gamma = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0 \text{ and } d > 0, \\ \mathbb{H} & \text{if } c = 0 \text{ and } d < 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

Values of the Shintani-Faddeev cocycle at $\tau = \frac{m}{n}$ and quantum modularity

Proposition (K 2022+)

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and let $m, n \in \mathbb{Z}$, $n > 0$ such that $\gcd(m, n) = 1$ and $c\frac{m}{n} + d > 0$. For $\mathrm{Im}(z) > 0$,

$$\sigma_\gamma\left(z, \frac{m}{n}\right) = \frac{D_{\mathrm{e}\left(\frac{am+bn}{cm+dn}\right)}\left(\mathrm{e}\left(\frac{nz}{cm+dn}\right)\right)^{-1/(cm+dn)}}{D_{\mathrm{e}\left(\frac{m}{n}\right)}\left(\mathrm{e}(z)\right)^{-1/n}},$$

$$\text{where } D_\zeta(w) = \prod_{k=1}^{n-1} (1 - \zeta^k w)^k \text{ for } \zeta = \mathrm{e}\left(\frac{m}{n}\right).$$

- $D_\zeta(w)$ is called the **cyclic quantum dilogarithm** and plays a role in Garoufalidis and Zagier's work on Nahm's conjecture.
- The proposition above **almost** implies that $f(z, \frac{m}{n}) = -\frac{1}{n} \log D_{\mathrm{e}\left(\frac{m}{n}\right)}(\mathrm{e}(z))$ defines a quantum Jacobi form of weight 0 in the sense of Bringmann and Folsom.

Hilbert's 12 problem and the Stark conjectures

- Kronecker-Weber theorem says that the abelian Galois extensions of \mathbb{Q} are generated by the values of $e(z) = e^{2\pi iz}$ at rational values of z .
- Hilbert's 12th problem asks for an extension of Kronecker-Weber where \mathbb{Q} is replaced by a general number field K .
- Given any K , Hilbert wanted analytic functions that play the role of $e(z)$.
- Harold Stark conjectured in a series of papers (1971–1980) that $\exp(cZ(1))$, for certain linear combinations $Z(s)$ of Hecke L -functions of K , generate abelian extensions of K .

L-functions at $s = 1$: rational example

The following formula can be proved using calculus. Try it!

Example

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + \dots + \dots = \frac{1}{\sqrt{2}} \log(1 + \sqrt{2})$$

The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. The right-hand side involves $\varepsilon = 1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.

L-functions at $s = 1$: imaginary quadratic example

The following formula is proved using the theory of complex multiplication for elliptic curves.

Example

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{e(m/5) - e(2m/5)}{m^2 + mn + n^2} = \frac{2\pi}{\sqrt{3}} \log \left(\varepsilon^{1/5} \right)$$

where $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65 + 29\sqrt{5})}$.

The left-hand side is a linear combination of Hecke L-values at $s = 1$ for $\mathbb{Q}(\sqrt{-3})$. The right-hand side involves an algebraic unit ε in the ray class field modulo (5) for $\mathbb{Q}(\sqrt{-3})$.

This example is related to the 5-torsion points of the CM elliptic curve $y^2 = x^3 + 1$. This elliptic curve has complex multiplication by $\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$.

L-functions at $s = 1$: real quadratic example

The following formula is an open conjecture!

Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n \in \mathbb{Z} \\ -\frac{5}{3}m \leq n < \frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where $\varepsilon \approx 3.890861714$ is a root of the polynomial equation

$$\begin{aligned} x^8 - (8 + 5\sqrt{3})x^7 + (53 + 30\sqrt{3})x^6 - (156 + 90\sqrt{3})x^5 \\ + (225 + 130\sqrt{3})x^4 - (156 + 90\sqrt{3})x^3 + (53 + 30\sqrt{3})x^2 \\ - (8 + 5\sqrt{3})x + 1 = 0. \end{aligned}$$

The number ε is an algebraic unit in the ray class field of $\mathbb{Q}(\sqrt{3})$ modulo $5\infty_2$. This conjecture is part of the Stark conjectures.

Ray class groups and ray class fields

Let K be a number field and \mathcal{O}_K its ring of integers. Let \mathfrak{m} be an ideal in \mathcal{O}_K , and let Σ be a subset of the real embeddings of K .

Definition (Ray class group modulo \mathfrak{m}, Σ)

$$\text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K) = \frac{\{\text{fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{m}\}}{\{a\mathcal{O}_K \text{ s.t. } a \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}}$$

Class field theory associates to $\text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$ a **ray class field** $H_{\mathfrak{m}, \Sigma}$, an abelian extension of K with Galois group $\text{Gal}(H_{\mathfrak{m}, \Sigma}/K) = \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$. Varying \mathfrak{m} and Σ , the ray class fields are cofinal among all abelian extensions of K .

Zeta functions associated to ray classes

Definition

For $\mathcal{A} \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$, the associated zeta function is

$$\zeta(s, \mathcal{A}) = \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in \mathcal{A}}} \text{Nm}(\mathfrak{a})^{-s}.$$

Let $R \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$ be the ideal class

$$\mathcal{R} = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}.$$

Definition

For $\mathcal{A} \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$, the associated differenced zeta function is

$$Z_{\mathcal{A}}(s) = \zeta(s, \mathcal{A}) - \zeta(s, \mathcal{R}\mathcal{A}).$$

Rank 1 abelian Stark conjecture over a real quadratic field

Conjecture (Stark, 1976, special case)

Setup:

- Let K be a real quadratic number field.
- Consider $0 \neq \mathfrak{m} \leq \mathcal{O}_K$ with the property that, if $\varepsilon \in \mathcal{O}_K^\times$ and $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$, then one of ε or $-\varepsilon$ is totally positive.
- Let \mathcal{A} be a ray ideal class in $\text{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K)$.
- Let $H_{\mathfrak{m}\infty_j}$ be the ray class field of K modulo $\mathfrak{m}\infty_j$.
- Let ρ_j be the real embedding of $H_{\mathfrak{m}\infty_1\infty_2}$ associated to ∞_j .

Then,

- (1) $Z'_{\mathcal{A}}(0) = \log(\rho_1(\varepsilon_{\mathcal{A}}))$ for a unit $\varepsilon_{\mathcal{A}} \in H_{\mathfrak{m}\infty_2}$.
- (2) The units $\varepsilon_{\mathcal{A}}$ are compatible with the isomorphism $\text{Art} : \text{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K) \rightarrow \text{Gal}(H_{\mathfrak{m}\infty_2}/K)$. Specifically, $\varepsilon_{\mathcal{A}} = \varepsilon_{\text{id}}^{\text{Art}(\mathcal{A})}$.

Hilbert's 12th problem and cocycles

- Hilbert's 12th problem was solved for imaginary quadratic fields using complex multiplication values of modular functions (0-cocycles for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$).
- **Hope:** Use “real multiplication values” of 1-cocycles to solve Hilbert's 12th problem for real quadratic fields.
- In fact, it's no longer just a hope: Darmon, Pozzi, and Vonk (2021) prove algebraicity results for real multiplication values of their p -adic Dedekind-Rademacher cocycle using the groundbreaking work of Dasgupta and Kakde (2021) on the Gross-Stark and Brumer-Stark conjectures.
- But what about the complex (not p -adic) setting?

Real multiplication values of a weight cocycle

- Let $w_\gamma(\tau)$ be a weight cocycle.
- Consider a real quadratic number β .
- Suppose $\gamma \in \Gamma$ is the “positive” generator for the stabilizer of β in Γ .
- Then, the values $w_\gamma(\beta)$ is the **real multiplication value** of w at β .

Real multiplication values of $\mathfrak{w}_\gamma^{\mathbf{p}}(\tau)$

Theorem (K 2022+)

Let K be a real quadratic field, $N \in \mathbb{N}$. Let $\mathcal{A} \in \text{Cl}(\mathcal{O}_K)$, choose some $\mathfrak{b} \in \mathcal{A}^{-1}$ coprime to $N\mathcal{O}_K$, and write $\mathfrak{b} = \mathbb{Z} + \beta\mathbb{Z}$ for some $\beta \in K$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the positive generator for $\text{stab}(\beta)$ in $\Gamma(N)$. Let $\beta_0 \in \mathcal{O}_K$ such that $\beta \equiv \beta_0 \pmod{N}$. For $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, let

$$\mathcal{A}_{\mathbf{p}} = \{\alpha\mathcal{O}_K : \alpha \equiv p_1\beta + p_2 \pmod{N} \text{ and } \rho_2(\alpha) > 0\} \in \widetilde{\text{Cl}}_{N\infty_2}(\mathcal{O}_K).$$

For an easily computable integer n ($= 1$ if $\beta_0\mathcal{O}_K + N\mathcal{O}_K = \mathcal{O}_K$),

$$\begin{aligned} & n \exp(Z'_{N\infty_2}(0, \mathfrak{b}^{-1}\mathcal{A}_{\mathbf{p}})) \\ &= \left(\zeta_8^{-1} \epsilon(\gamma) e\left(\frac{N+1}{4} (bp_1^2 - (a-d)p_1p_2 - cp_2^2) \right) \mathfrak{w}_\gamma^{N-1} \mathbf{p}(\beta) \right)^2. \end{aligned}$$

Proof outline

- Use Jacobi triple product and modular properties of $\vartheta_1(z, \tau)$ and $\eta(\tau)$ to establish a relation between $\mathfrak{w}_\gamma^r(\tau)$ and $\mathfrak{w}_\gamma^{-r}(\tau)$.
- Write Tangedal's version of Shintani's formula (involving the Hirzebruch-Jung continued fraction of β) in terms of the Shintani-Faddeev cocycle, and use the cocycle condition to "telescope" the product.
- One is left with a complicated-looking root of unity factor...
- ...that may be simplified greatly using the combinatorics of continued fraction expansions and the relation between $\mathfrak{w}_\gamma^r(\beta)$ and $\mathfrak{w}_\gamma^{-r}(\beta)$ previously established.
- The maximal order \mathcal{O}_K may also be replaced by an arbitrary order \mathcal{O} .

Future directions

- Big picture: Want to prove an algebraicity result for $\mathfrak{w}_\gamma^r(\beta)$.
- Work out (conjecturally, at least) the action of the Artin map on the sign under the square root.
- Understand cohomology group containing $\gamma \mapsto \mathfrak{w}_\gamma^r(\tau)$, that is, $H^1_{\tilde{\mathcal{B}}}(\Gamma(N), \mathcal{M}_{\mathbb{H}}^\times)$ with $\tilde{\mathcal{B}}_\gamma = \mathcal{M}_{\tilde{D}_\gamma}^\times$ (or a smaller group, if this turns out to be too big).
- Connect complex and p -adic cocycles.
- Try to do something with formulas and perspectives originating in conformal field theory literature. Connect CFT to CFT?

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Interlude: rational values
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Stark conjectures
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Real multiplication values
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Thank you!

Thank you for listening! Any questions?