Analytic Number Theory: Problem Set 3
Due November 10, 2022, in class

Corrections and clarifications are shown in red.

(1) [4 points] Let G_1 and G_2 be finite abelian groups. Prove the isomorphism of character groups

$$\widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}.$$

(2) If $a \in \mathbb{Z}/m\mathbb{Z}$ and χ is a Dirichlet character modulo m, define the *Gauss sum*

$$g_a(\chi) = \sum_{u \in \mathbb{Z}/m\mathbb{Z}} \chi(u)\, \mathrm{e}\left(\frac{au}{m}\right),$$

where $\mathrm{e}(z) = e^{2\pi i z}$.

(a) [3 points] If $a \in (\mathbb{Z}/m\mathbb{Z})^\times$, prove that $g_a(\chi) = \overline{\chi(a)} g_1(\chi)$.

(b) [5 points] If $m = p$ is a prime number and χ is nontrivial, prove that $|g_1(\chi)| = \sqrt{p}$. (Hint: $|g_1(\chi)|^2 = g_1(\chi)\overline{g_1(\chi)}$.)

(3) A natural number n is called *squarefull* if $p|n$ implies $p^2|n$. Define the sets

$$\mathcal{S} = \{n \in \mathbb{N} : n \text{ is squarefull}\};$$
$$\mathcal{S}(x) = \{n \in \mathbb{N} : n \text{ is squarefull and } n \le x\}.$$

Let $S(x) = |\mathcal{S}(x)|$, the number of squarefull numbers up to x. For this problem, you may need to use the results on squarefree numbers given on pages 36 and 37 of Montgomery and Vaughan, specifically Theorem 2.2 and the discussion that follows.

(a) [4 points] Show that each $n \in \mathcal{S}$ can be written in a unique was as $n = m^3 d^2$ such that m is squarefree. Use this fact to express the Dirichlet series

$$F_{\text{full}}(s) := \sum_{n \in \mathcal{S}} \frac{1}{n^s}$$

in terms of the Riemann zeta function.

(b) [4 points] Estimate the sum

$$T(x) := \sum_{n \in \mathcal{S}(x)} \sqrt{\frac{x}{n}},$$

and use this to deduce that

$$S(x) = O\left(\sqrt{x}\log(x)\right).$$

(c) [4 points] Use (a) and (b) (or another method) to prove that

$$S(x) \sim \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\sqrt{x} \text{ as } x \to \infty.$$

If you can show (c) without using (b), you may deduce part (b) from (c) without estimating $T(x)$.

(4) Let $0 < \alpha \leq 1$, and for $\mathrm{Re}(s) > 1$, define the *Hurwitz zeta function* by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

(a) [4 points] Show that, for $\mathrm{Re}(s) > 1$,

$$\Gamma(s)\zeta(s,\alpha) = \int_0^{\infty} \frac{e^{-\alpha x}}{1-e^{-x}} x^s \frac{dx}{x}.$$

(b) [2 points] Suppose that $0 < r < 2\pi$, and define the contour integral

$$I_r(s,\alpha) = \frac{1}{2\pi i} \int_{\mathcal{C}(r)} \frac{e^{\alpha z}}{1-e^z} z^s \frac{dz}{z},$$

where $\mathcal{C}(r)$ is a contour that follows a horizontal line to the right from $-\infty - ri$ to $-ri$, then follows a counterclockwise semicircle from $-ri$ through r to ri, then follows a horizontal line to the left from ri to $-\infty + ri$. Show that $I_r(s,\alpha)$ is independent of r for $0 < r < 2\pi$.

(c) [2 points] Show that $I_r(s,\alpha)$ defines an entire function of s.

(d) [3 points] By letting $r \to 0$, show that, for $\mathrm{Re}(s) > 1$,

$$\zeta(s,\alpha) = \Gamma(1-s) I_r(s,\alpha)$$

(e) [3 points] Deduce that $\zeta(s,\alpha)$ has an analytic continuation to the whole complex plane except at $s = 1$, where it has a simple pole at $s = 1$ with residue 1.

(f) [3 points] Let χ be a Dirichlet character modulo m. Prove that

$$L(s,\chi) = \frac{1}{m^s} \sum_{a=1}^{m} \chi(a) \zeta\left(s, \frac{a}{m}\right).$$

(5) In this problem, you may assume the following product formula for the gamma function:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Here, γ is Euler's constant.

(a) [3 points] Prove the formula

$$\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(1)}{\Gamma(1)} = -\sum_{n=0}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n+1}\right).$$

(b) [2 points] Show that $\Gamma'(1) = -\gamma$.

(c) [4 points] Evaluate the limit

$$\lim_{s \to 1^+} \left(\zeta(s,\alpha) - \zeta(s)\right).$$

By doing so, compute the constant term of the Laurent expansion of the Hurwitz zeta function at $s = 1$, showing that

$$\zeta(s,\alpha) = \frac{1}{s-1} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + O(s-1) \text{ as } s \to 1.$$

(6*) [up to 10 points extra credit] This problem is not officially assigned, but correct solutions (to all or some parts) will earn extra points toward the total score on Problem Sets 1–3. This problem uses the Gauss sums introduced in problem (3) to prove the quadratic reciprocity law. Part (c) requires some algebraic number theory.

(a) For an odd prime number p, define the *quadratic character* or *Legendre symbol* by

$$\chi_p(a) := \left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } p \nmid a \text{ and } a \equiv b^2 \pmod{p} \text{ for some } b \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } a \not\equiv b^2 \pmod{p} \text{ for any } b \in \mathbb{Z}, \\ 0 & \text{if } p \mid a. \end{cases}$$

Explain why χ_p is a Dirichlet character modulo p, and prove that $\chi(a) \equiv a^{\frac{p-1}{2}} \pmod{p}$. (You may use the fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group.)

(b) For an odd prime p, prove that $\overline{g_1(\chi_p)} = g_{-1}(\chi_p)$ and thus $g_1(\chi_p)^2 = (-1)^{\frac{p-1}{2}} p$.

(c) Let p and q be distinct odd primes, and let $p^* = (-1)^{\frac{p-1}{2}} p$. Use (a) and (b) (or another method) to show that

$$(g_1(\chi_p))^{q-1} \equiv \chi_q(p^*) \pmod{q}$$

Also, interpreting the following congruence in the ring $\mathbb{Z}[\zeta_p]$ where ζ_p is a primitive p-th root of unity, show (directly or through other means) that

$$(g_1(\chi_p))^q \equiv g_q(\chi_p) \pmod{q}$$

Deduce the *quadratic reciprocity law*:

$$\chi_p(q)\chi_q(p) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$