SIC-POVMs from binary quadratic forms

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ICERM Workshop on Recent Progress on Optimal Point Distributions and Related Fields

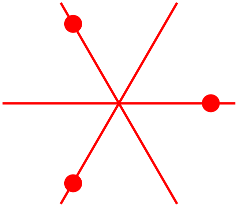
June 4, 2024

This talk will discuss:

- Background on SICs and related objects
- Results/conjectures of K and Lagarias [2024] on counting SICs and fields generated by SICs
- Results/conjectures of Appleby, Flammia, and K [2024+] on ghostly constructions giving a conjectural construction of SICs in every dimension (among other things)

Complex equiangular lines

You can draw three equiangular lines through the origin in $\mathbb{R}^2$:



Definition

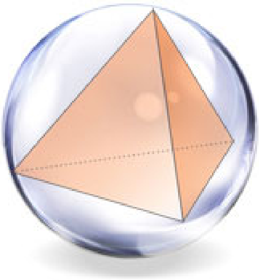
A finite set $S = \{\mathbb{C}v_1, \ldots, \mathbb{C}v_n\} \subset \mathbb{P}^{d-1}(\mathbb{C})$ with $\| v_j \| = 1$ is equiangular if $|\langle v_j, v_k \rangle| := |\overline{v}_j \cdot v_k| = \alpha$ for $j \neq k$.

In terms of the Hermitian projections onto the lines, equiangularity means $\mathrm{Tr}(\Pi_j \Pi_k) = \alpha^2$ for $j \neq k$.

Is it possible to find more than three equiangular lines in $\mathbb{C}^2$?

Complex equiangular lines

Yes! Take $S = \left\{[1 : \frac{1+i}{1+\sqrt{3}}], [1 : \frac{-1-i}{1+\sqrt{3}}], [\frac{1+i}{1+\sqrt{3}} : 1], [\frac{-1-i}{1+\sqrt{3}} : 1]\right\}$.



Proposition (Delsarte, Goethals, and Seidel; 1975)

Consider a set $S \subset \mathbb{P}^{d-1}(\mathbb{C})$ of n equiangular lines of common angle $\arccos(\alpha)$. Then, $n \leq d^2$. If $n = d^2$, then $\alpha = \frac{1}{\sqrt{d+1}}$.

SICs

Definition (SIC)

A SIC (SIC-POVM; symmetric informationally-complete positive operator-valued measure) is a set $\mathcal{S} = \{\Pi_1, \ldots, \Pi_{d^2}\} \subset M_{d\times d}(\mathbb{C})$ such that:

(1) $\Pi_j^2 = \Pi_j$

(2) $\mathrm{Tr}(\Pi_j \Pi_k) = \frac{1}{d+1}$ for $j \neq k$

(3) $\mathrm{rk}\, \Pi_j = 1$

(4) $\Pi_j^\dagger = \Pi_j$

(We dispense with the normalization factor $\frac{1}{d}$ for convenience.)

A SIC is equivalent to:

- A maximal set of complex equiangular lines.
- A maximal complex equiangular tight frame (ETF).
- A minimal complex projective 2-design.

r-SICs

If we generalize from equiangular lines to equichordal subspaces, Delsarte, Goethals, and Seidel's upper bound of d^2 still holds.

Definition (r-SIC)

An r-SIC is a set $\mathcal{S} = \{\Pi_1, \ldots, \Pi_{d^2}\} \subset M_{d \times d}(\mathbb{C})$ such that:

(1) $\Pi_j^2 = \Pi_j$

(2) $\mathrm{Tr}(\Pi_j \Pi_k) = \frac{r(dr-1)}{d^2-1}$ for $j \neq k$

(3) $\mathrm{rk}\, \Pi_j = r$

(4) $\Pi_j^\dagger = \Pi_j$

An r-SIC is equivalent to a maximal ECTFF (equichordal tight fusion frame). ECTFFs have also been called STFFs (symmetric tight fusion frames).

Weak Zauner's Conjecture

Conjecture (Zauner 1999)

There is at least one SIC in every dimension $d \geq 1$.

SICs are known from exact solutions in dimensions 1–53 and many other dimensions as large as $d = 1299$.

Numerical (probable) SICs are known in dimension 1–193 and various higher dimensions. Most exact and numerical solutions have been found by Grassl and Scott.

By studying known SICs, people (starting with Zauner) have formulated increasingly more precise refinements of this conjecture.

The Weyl–Heisenberg group

Let $\zeta = e^{\frac{2\pi i}{d}}$ and $\xi = -e^{\frac{\pi i}{d}}$. The Weyl–Heisenberg group (associated to $\mathbb{Z}/d\mathbb{Z}$) is the finite unitary matrix group

$$\mathrm{WH}(d) = \{\xi^k X^{p_1} Z^{p_2} : k, p_1, p_2 \in \mathbb{Z}\}, \text{ where}$$

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & 0 & \zeta^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta^{d-1} \end{pmatrix}.$$

A special set of coset representatives for WH(d) modulo its center is

$$D_{\mathbf{p}} = \xi^{p_1 p_2} X^{p_1} Z^{p_2}$$

for $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/d\mathbb{Z})^2$.

Strong Zauner's Conjecture

Conjecture (Zauner 1999)

There is at least one SIC $\mathcal{S}$ in every dimension $d \geq 1$ satisfying the following properties:

(1) $\mathcal{S}$ is Weyl–Heisenberg covariant (a.k.a., a WH-SIC): $\mathcal{S} = \{D_{\mathbf{p}}^{-1} \Pi D_{\mathbf{p}}\}_{\mathbf{p} \in (\mathbb{Z}/d\mathbb{Z})^2}$ for some fiducial projector Π.

(2) $U_{\mathrm{Zau}}^{-1} \Pi U_{\mathrm{Zau}} = \Pi$ for the order three unitary Zauner matrix with entries

$$(U_{\mathrm{Zau}})_{k\ell} = \frac{1}{\sqrt{d}} \, \mathrm{e}\left(\frac{d-1}{24} + \frac{2k\ell + (d+1)\ell^2}{2d} \right).$$

Here, $\mathrm{e}(z) = e^{2\pi i z}$.

All but one of the known SICs satisfy (1) (up to unitary equivalence). Some WH-SICs do not satsfy (2).

Geometric equivalence of r-SICs

An r-SIC remains an r-SIC after a unitary “rotation” or an anti-unitary “reflection”—forming the extended unitary group $\mathrm{EU}(d)$.

We consider two r-SICs $\{\Pi_j\}$ and $\{\Pi'_j\}$ to be geometrically equivalent if there is some $U \in \mathrm{EU}(d)$ such that $U^{-1}\Pi_j U = \Pi'_j$.

Proposition

WH-r-SICs with fiducial projectors Π and Π' are geometrically equivalent if and only if $U^{-1}\Pi U = \Pi'$ for some $U \in \mathrm{EC}(d)$, where

$$\mathrm{EC}(d) = \{U \in \mathrm{EU}(d) : U^{-1}HU \in \mathrm{WH}(d) \text{ for all } H \in \mathrm{WH}(d)\}$$

is a finite group.

Definition

The $\mathrm{EC}(d)$-orbit of a WH-r-SIC is $[\mathcal{S}] := \{U^{-1}\mathcal{S}U : U \in \mathrm{EC}(d)\}$.

A known infinite family of r-SICs

Unlike 1-SICs, r-SICs are known to exist in arbitrarily large dimension.

Proposition (Appleby, Bengtsson, Flammia, and Goyeneche 2019; Construction of Wigner $\frac{d\pm1}{2}$-SICs)

For any $d \in \mathbb{N}$, the matrices

$$\Pi^{\pm} = \frac{1}{2}(I \pm U_P) \text{ with } U_P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}$$

are fiducial projectors for WH-r-SICs of rank $r = \frac{d\pm1}{2}$.

As with SICs in dimension $d = 3$, there is expected to be a continuous family of WH-$\frac{d\pm1}{2}$-SICs, with the Wigner MEFF generalizing the Hasse SIC.

Past work on SICs and number theory

- [Appleby, Yadsan-Appleby, Zauner 2012] Galois groups of known SICs are solvable; conjectures about Galois group structure
- [Appleby, Flammia, McConnell, Yard 2016] SIC fields are often ray class fields of real quadratic field $\mathbb{Q}(\sqrt{(d+1)(d-3)})$
- [K thesis 2017] Connected SIC in $d = 5$ to Stark units; some conjectures about counting SICs
- [K 2019] Conjectural construction of SICs from Stark units in prime dimension $d \equiv 5 \pmod 6$
- [Appleby, Bengtsson, Grassl, Harrison, McConnell 2021] Conjectural construction of SICs from Stark units, prime $d = n^2 + 3$
- [Bengtsson, Grassl, McConnell 2024] Conjectural construction of SICs from Stark units, $d = 4p = n^2 + 3$, p prime

Class groups of binary quadratic forms

Let $\Delta \equiv 0, 1 \pmod 4$ be a nonsquare integer.

$$\mathcal{Q}(\Delta) := \{Q(x,y) = ax^2 + bxy + cy^2 : b^2 - 4ac = \Delta\}.$$
$$\mathcal{Q}_{\mathrm{prim}}(\Delta) := \{Q(x,y) \in \mathcal{Q}(\Delta) : \gcd(a,b,c) = 1\}.$$

If $\Delta = f^2\Delta_0$ for Δ_0 a fundamental discriminant, then

$$\mathcal{Q}(\Delta) = \bigsqcup_{f'|f} \left\{ \frac{f}{f'} Q(x,y) : Q \in \mathcal{Q}_{\mathrm{prim}}((f')^2\Delta) \right\}.$$

The group $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathcal{Q}(\Delta)$ and $\mathcal{Q}_{\mathrm{prim}}(\Delta)$ by the twisted action

$$(Q|_A)(x,y) = (\det A)\, Q(\alpha x + \beta y, \gamma x + \delta y)$$

for $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$.

Class groups of binary quadratic forms

Theorem (Gauss 1801)

The set of equivalence class of the group action

$$\mathrm{Cl}(\mathcal{O}_\Delta) := \mathcal{Q}_{\mathrm{prim}}(\Delta)/\,\mathrm{GL}_2(\mathbb{Z})$$

is a finite set. Moreover, it has the structure of a group with operation $[Q_1] * [Q_2] = [Q_3]$ defined uniquely by the condition

$$\begin{aligned} Q_3(X, Y) &= Q_1(x_1, y_1)Q_2(x_2, y_2) \text{ for some} \\ X &= px_1y_1 + qx_1y_2 + ry_1x_2 + sy_1x_2, \\ Y &= p'x_1y_1 + q'x_1y_2 + r'y_1x_2 + s'y_1x_2, \end{aligned}$$

with $p, q, r, s, p', q', r', s' \in \mathbb{Z}$.

On non-primitive forms, Gauss composition defines a monoid

$$\mathrm{Clm}(\mathcal{O}_\Delta) := \mathcal{Q}(\Delta)/\,\mathrm{GL}_2(\mathbb{Z}) \longleftrightarrow \bigsqcup_{f'|f} \mathrm{Cl}(\mathcal{O}_{(f')^2\Delta_0}).$$

Counting SICs

Conjecture (K 2017)

Fix $d \neq 3$, and let $\Delta = (d+1)(d-3)$.Then,

$$|\text{WH-SIC}(d)/\operatorname{EC}(d)| = |\operatorname{Clm}(\mathcal{O}_\Delta)|\,,$$

where $\operatorname{Clm}(\mathcal{O}_\Delta) = \mathcal{Q}(\Delta)/\operatorname{GL}_2(\mathbb{Z})$ is the set of twisted $\operatorname{GL}_2(\mathbb{Z})$-classes of binary quadratic forms of discriminant Δ.

This conjecture has been verified for $d \leq 90$, assuming Scott and Grassl's tables of SICs are complete (K and Lagarias 2024).

The quantity $|\operatorname{Clm}(\mathcal{O}_\Delta)|$ is also the number of $\operatorname{GL}_2(\mathbb{Z})$-conjugacy classes of elements of $\operatorname{SL}_2(\mathbb{Z})$ of trace $d-1$. (This formulation removes Δ from the picture.)

Number of SICs in dimension d

The size of the class monoid $|\mathrm{Clm}(\mathcal{O}_\Delta)|$ is a sum of class numbers, whose sizes can be estimated from values of L-functions.

We know $|\mathrm{Clm}(\mathcal{O}_\Delta)| \to \infty$ as $d \to \infty$. In particular...

Theorem (Byeon, Kim, and Lee 2007)

For $\Delta = (d+1)(d-3)$, $|\mathrm{Clm}(\mathcal{O}_\Delta)| = 1$ if and only if $d \in \{1, 2, 4, 5, 6, 10, 22\}$.

So there should be more than one SIC in every dimension $d > 22$.

Theorem (K and Lagarias 2024)

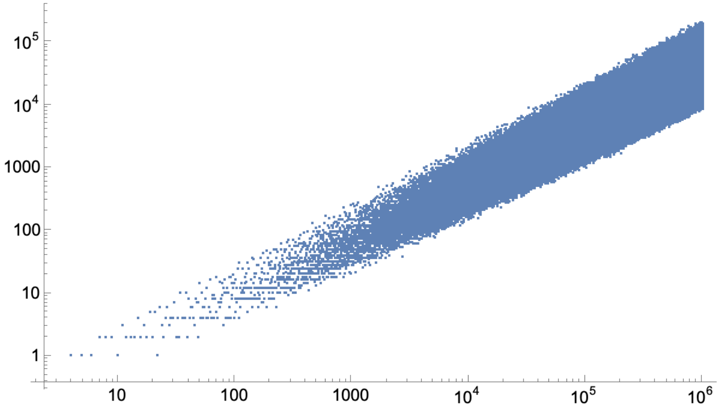
For $\Delta = (d+1)(d-3)$, as $d \to \infty$,

$$\log |\mathrm{Clm}(\mathcal{O}_\Delta)| = \log(d) + o(\log(d))$$

The above result is proven using the Brauer–Siegel theorem, with an ineffective implicit constant.

Number of SICs in dimension d

Plot by Steven Flammia; assumes conjectures.



Galois multiplets of SICs

Let $\sigma_c : \mathbb{C} \to \mathbb{C}$ denote complex conjugation $\sigma_c(z) = \overline{z}$. Consider the centralizer

$$\mathrm{C}_{\mathrm{Gal}(\mathbb{C}/\mathbb{Q})}(\sigma_c) = \left\{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) : (\forall z \in \mathbb{C})\, \sigma(\overline{z}) = \overline{\sigma(z)}\right\}.$$

Lemma (easy)

If $\mathcal{S} = \{\Pi_1, \ldots, \Pi_{d^2}\}$ is an r-SIC and $\sigma \in \mathrm{C}_{\mathrm{Gal}(\mathbb{C}/\mathbb{Q})}(\sigma_c)$, then $\sigma(\mathcal{S}) = \{\sigma(\Pi_1), \ldots, \sigma(\Pi_{d^2})\}$ is a r-SIC.

Definition

The multiplet of a SIC $\mathcal{S}$ is the set of EC(d)-orbits

$$[[\mathcal{S}]] = \left\{[\sigma(\mathcal{S})] : \sigma \in \mathrm{C}_{\mathrm{Gal}(\mathbb{C}/\mathbb{Q})}(\sigma_c)\right\}.$$

(It is a finite set if and only if $\mathcal{S}$ is algebraic.)

Counting SICs by multiplets

Conjecture (K 2017; K and Lagarias 2024)

Let $d \neq 3$ and $\Delta = (d+1)(d-3)$, and write $\Delta = f^2\Delta_0$. Then there is a bijection (not explicitly constructed)

$$\#\{\text{pos. divisors of } f\} \to \#\{[[\mathcal{S}]] : \mathcal{S} \ d\text{-dim'l WH-SIC}\}$$
$$f' \mapsto [[\mathcal{S}_{f'}]].$$

The size of each multiplet is

$$\#[[\mathcal{S}_{f'}]] = \#\,\mathrm{Cl}(\mathcal{O}_{(f')^2\Delta_0}).$$

This conjecture has been verified for $d \leq 90$, assuming that Scott and Grassl's tables are complete.

Definition of fields generated by general SICs (makes sense for r-SICs)

"The field generated by a SIC" or "the SIC field" has been used in several inequivalent ways in the literature.

Here are two associated to $\mathcal{S} = \{\Pi_1, \ldots, \Pi_{d^2}\}$:

- projector SIC field:

$$F^{\text{proj}}(\mathcal{S}) := \mathbb{Q}((\Pi_k)_{ij} : 1 \le k \le d^2, 1 \le i, j \le d).$$

- triple product SIC field:

$$F^{\text{trip}}(\mathcal{S}) := \mathbb{Q}(\text{Tr}(\Pi_{k_1}\Pi_{k_2}\Pi_{k_3}) : 1 \le k_1, k_2, k_3 \le d^2).$$

Proposition

If one defined the unitary invariant SIC field to be

$$F^{\text{inv}}(\mathcal{S}) := \mathbb{Q}(\text{Tr}(\Pi_{k_1} \cdots \Pi_{k_m}) : 1 \le k_1, \ldots, k_m \le d^2),$$

then $F^{\text{inv}}(\mathcal{S}) = F^{\text{trip}}(\mathcal{S})$.

Proof of $F^{\mathrm{inv}}(\mathcal{S}) = F^{\mathrm{trip}}(\mathcal{S})$ for r-SICs

Proof.

The $\{\Pi_k\}$ form a basis for $M_d(\mathbb{C})$, so there are "structure constants"

$$\Pi_{k_1}\Pi_{k_2} = \sum_\ell \alpha^\ell_{k_1 k_2}\Pi_\ell. \tag{1}$$

Taking traces div. by r, $\sum_\ell \alpha^\ell_{k_1 k_2} = \frac{d^2+dr}{d^2-1}\delta_{k_1 k_2} + \frac{dr-1}{d^2-1}$. Moreover,

$$\begin{aligned}\mathrm{Tr}(\Pi_{k_1}\Pi_{k_2}\Pi_{k_3}) &= \sum_\ell \alpha^\ell_{k_1 k_2}\mathrm{Tr}(\Pi_\ell \Pi_{k_3}) \\ &= \frac{dr}{d+1}\alpha^{k_3}_{k_1 k_2} + \frac{dr(d+r)(dr-1)}{(d^2-1)^2}\delta_{k_1 k_2} + \frac{r(dr-1)^2}{d^2-1}. \end{aligned} \tag{2}$$

Now, solve (2) for the structure constants $\alpha^{k_3}_{k_1 k_2}$ in terms of the triple products. Note that you can use (1) repeatedly to express $\mathrm{Tr}(\Pi_{k_1}\cdots\Pi_{k_m})$ as a polynomial of the structure constants. □

Hilbert's 12th problem and explicit class field theory

Hilbert's 12th problem asks for an explicit construction of abelian Galois extensions of number fields using special values of transcendental functions, preferable with a geometric interpretation.

The abelian extensions of $\mathbb{Q}$ are cyclotomic fields—generated by the values of $\mathrm{e}(z) = e^{2\pi iz}$ at rational numbers.

Those of an imaginary quadratic field K are generated by "complex multiplication (CM) values" of modular functions.

Both have geometric interpretations: Torsion points on the unit circle (for $\mathbb{Q}$) and on a CM elliptic curve (for K).

A geometric interpretation for abelian extensions of a real quadratic field appears to come from SICs and r-SICs!

Class field theory (or orders of number fields)

Consider a number field K and data $(\mathcal{O}; \mathfrak{m}, \Sigma)$ with

- $\mathcal{O}$ a subring of K with abelian group structure $\mathbb{Z}^{[K:\mathbb{Q}]}$ (an order),
- $\mathfrak{m}$ a nonzero ideal of $\mathcal{O}$, and
- Σ a subset of the finite set of embeddings $\{K \to \mathbb{R}\}$.

Then one can abstractly define

- a finite abelian group $\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$, the ray class group, and
- a number field $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$, the ray class field, with $\mathrm{Gal}(H^{\mathcal{O}}_{\mathfrak{m},\Sigma}/K \cong \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$,
- such that, fixing $\mathcal{O}$ and varying $(\mathfrak{m}, \Sigma)$, every abelian extension of K is contained in some $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$.

In the case of the maximal order, existence of the ray class field is due to Takagi (1920). The non-maximal order case is due to K and Lagarias (2022), building on class field theory.

Ray class fields for quadratic orders

The orders is the quadratic field $\mathbb{Q}(\Delta_0)$ for a fundamental discrimiant Δ_0 are

$$\mathcal{O}_{f^2\Delta_0} = \mathbb{Z}\left[f\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right] = f\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\mathbb{Z} + \mathbb{Z}.$$

For $\Delta = (d+1)(d-3) = f^2\Delta_0$, we define some ray class fields of interest for $f'|f$:

$$E_{d,f'} = H^{\mathcal{O}'}_{d\mathcal{O}',\{\infty_1,\infty_2\}} \text{ with } \mathcal{O}' = \mathcal{O}_{(f')^2\Delta_0},$$
$$E'_{d,f'} = H^{\mathcal{O}'}_{d'\mathcal{O}',\{\infty_1,\infty_2\}} \text{ with } \mathcal{O}' = \mathcal{O}_{(f')^2\Delta_0},$$

where $d' = d$ if d is odd and $d' = 2d$ if d is even.

Clearly $E'_{d,f'} = E_{d,f'}$ if d is odd. One can show $[E'_{d,f'} : E_{d,f'}] = 2$ if d is even.

Conjecture about fields generated by SICs

Conjecture (K and Lagarias 2024)

Fix $d \neq 3$, and let $\Delta = (d+1)(d-3) = f^2\Delta_0$ with Δ_0 a fundamental discriminant. There is a bijective map (not explicitly constructed)

$$\mathcal{Q}(\Delta)/\operatorname{GL}_2(\mathbb{Z}) \to \text{WH-SIC}(d)/\operatorname{PEC}(d)$$
$$[Q] \mapsto [\mathcal{S}_Q]$$

having the following properties:

- If $Q(x,y) = \frac{f}{f'}(ax^2 + bxy + cy^2)$ with $\gcd(a,b,c) = 1$ and $f'|f$:
  - $F^{\text{trip}}(\mathcal{S}_Q) = E_{d,f'}$, and
  - $F^{\text{proj}}(\mathcal{S}_Q) = E'_{d,f'}$.
- The $[[\mathcal{S}_{Q_1}]] = [[\mathcal{S}_{Q_2}]]$ if and only if the associated $f'_1 = f'_2$.

Numerical evidence on fields generated by SICs

d	Δ_0	f'	$h_{\mathcal{O}}$	$[E_{d,f'} : K]$	$[E'_{d,f'} : K]$	multiplet	$[F^{\text{trip}}(\mathcal{S}) : K]$	$[F^{\text{proj}}(\mathcal{S}) : K]$
4	5	1	1	4	8	4a	4	8
5	12	1	1	16	16	5a	16	16
6	21	1	1	12	24	6a	12	24
7	8	1	1	12	12	7b	12	12
		2	1	24	24	7a	24	24
8	5	1	1	8	16	8b	8	16
		3	1	32	64	8a	32	64
9	60	1	2	72	72	9ab	72	72
10	77	1	1	48	96	10a	48	96
11	24	1	1	80	80	11c	$*$	80
		2	2	160	160	11ab	$*$	160
12	13	1	1	16	32	12b	$*$	32
		3	1	48	96	12a	$*$	96
13	140	1	2	192	192	13ab	$*$	192
14	165	1	2	144	288	14ab	$*$	288
15	8	1	1	48	48	15d	$*$	48
		2	1	96	96	15b	$*$	96
		4	2	192	192	15ac	$*$	192

Table from [K and Lagarias 2024]. Entries denoted by $*$ have not been computed.

Degeneration of SIC fields

Marcus Grassl observed (private communication, 2022) that, in dimensions $d \in \{47, 67, 259\}$, there are two multiplets generating the same field $F^{\mathrm{proj}}(\mathcal{S})$.

Theorem (K and Lagarias 2024)

We have $E_{d,f'} = E_{d,f''}$ (equivalently $E'_{d,f'} = E'_{d,f''}$) for $f'|f''$ and $f' \neq f''$ if and only if all of the following hold:

- $f'' = 2f'$,
- f' is odd, and
- $\mathrm{sqfreepart}((d+1)(d-3)) \equiv 1 \pmod 8$.

The set of d for which this occurs has asymptotic density

$$\lim_{X\to\infty} \frac{\{d \leq X : \mathrm{sqfreepart}((d+1)(d-3)) \equiv 1 \pmod 8\}}{X} = \frac{1}{48}.$$

Degen. for $d \in \{47, 67, 83, 175, 211, 259, 303, 339, 431, 447, 467, \ldots\}$

Ghost r-SICs

Definition (Ghost r-SICs)

A ghost r-SIC is a set $\{\Phi_1, \ldots, \Phi_{d^2}\} \subset M_{d\times d}(\mathbb{C})$ such that:

(1) $\Phi_j^2 = \Phi_j$

(2) $\mathrm{Tr}(\Phi_j \Phi_k) = \frac{r(dr-1)}{d^2-1}$ for $j \neq k$

(3) $\mathrm{rk}\, \Phi_j = r$

(4) $\Phi_j^\dagger = U_P \Phi_j$ (Parity-Hermitian) where

$$U_P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}$$

The Shintani–Faddeev modular cocycle

Theorem (K 2024+)

Let $\mathbf{r} = (r_1, r_2) \in \frac{1}{d}\mathbb{Z}^2$ and $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $A\mathbf{r} - \mathbf{r} \in \mathbb{Z}^2$, and (using the notation $\mathrm{e}(z) = e^{2\pi i z}$) let

$$\varpi^{\mathbf{r}}_A(\tau) = \prod_{k=0}^{\infty} \frac{1 - \mathrm{e}\left((k + r_2)\frac{\alpha\tau+\beta}{\gamma\tau+\delta} - r_1\right)}{1 - \mathrm{e}((k + r_2)\tau - r_1)} \text{ for } \mathrm{Re}(\tau) > 0.$$

Then $\varpi^{\mathbf{r}}_A(\tau)$ meromorphically continues to

$$\tau \in U_A = \mathbb{C} \setminus \{\tau \in \mathbb{R} : \gamma\tau + \delta \leq 0\},$$

with poles only at certain rational numbers.

Definition (K 2024+)

The map $A \mapsto \varpi^{\mathbf{r}}_A(\tau)$ is the Shintani–Faddeev modular cocycle.

Conjectural construction of ghost r-SICs and ("living") r-SICs

Conjecture (Appleby, Flammia, and K 2024+)

Let $d, r \in \mathbb{N}$ with $0 < r < \frac{d-1}{2}$ such that $n = \frac{d^2-1}{r(d-r)} \in \mathbb{Z}$; let $\Delta = n(n-4)$. Let $Q(x, y) = ax^2 + bxy + cy^2 \in \mathcal{Q}(\Delta)$, β a root of $Q(\beta, 1) = 0$. Let $B = \begin{pmatrix} (-b+n-2)/2 & -c \\ a & (b-n+2)/2 \end{pmatrix}$, and take the smallest positive power $A = B^k \equiv I \pmod{d}$. For $\mathbf{p} \in \mathbb{Z}^2/d\mathbb{Z}^2$, set

$$\nu_{\mathbf{p}} = \begin{cases} r, & \text{if } p_1 = p_2 = 0; \\ -\frac{1}{\sqrt{n}}\, \mathrm{e}\!\left(\frac{m(A,\mathbf{p})}{24d}\right) \varpi_A^{d^{-1}\mathbf{p}}(\beta)\,, & \text{else}, \end{cases}$$

with $m(A, \mathbf{p})$ an explicit, easily computable integer.

$$\Phi = \frac{1}{d} \sum_{\mathbf{p} \in (\mathbb{Z}/d\mathbb{Z})^2} \nu_{G\mathbf{p}} D_{\mathbf{p}}; \quad G = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$ Then,

(1) Φ is a fiducial projection of a WH covariant ghost r-SIC.

(2) $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta} \Rightarrow \sigma(\Phi)$ a fid. proj. of a WH-r-SIC.

Comments on the conjectural construction

- The conjecture accounts for all known SICs except those in $d = 3$ and the Hoggar lines in $d = 8$.
- The conjecture has been partially verified for all $d \leq 50$ and fully verified for the four SICs in dimension $d = 100$ (of which only one was previously known).
- The ghost r-SIC construction (1) would follow from a (possibly very difficult!) special value identity for $\mathfrak{w}^{\mathbf{r}}$ we call the Twisted Convolution Conjecture.
- The r-SIC construction (2) follows from the ghost construction under the Stark conjectures or a mild refinement.
- Look for Steven Flammia's Julia package `Zauner` coming soon to GitHub (as well as the preprints, coming soon to arXiv)

Some parameters of r-SICs with $r > 1$ (according to the conjecture)

d	r	K	d	r	K	d	r	K
11	3	$\mathbb{Q}(\sqrt{5})$	109	10	$\mathbb{Q}(\sqrt{6})$	271	16	$\mathbb{Q}(\sqrt{7})$
19	4	$\mathbb{Q}(\sqrt{3})$	131	11	$\mathbb{Q}(\sqrt{13})$	305	17	$\mathbb{Q}(\sqrt{285})$
29	5	$\mathbb{Q}(\sqrt{21})$	139	24	$\mathbb{Q}(\sqrt{21})$	341	18	$\mathbb{Q}(\sqrt{5})$
	8	$\mathbb{Q}(\sqrt{5})$	155	12	$\mathbb{Q}(\sqrt{35})$	377	48	$\mathbb{Q}(\sqrt{5})$
41	6	$\mathbb{Q}(\sqrt{2})$	181	13	$\mathbb{Q}(\sqrt{165})$	379	19	$\mathbb{Q}(\sqrt{357})$
55	7	$\mathbb{Q}(\sqrt{5})$	199	55	$\mathbb{Q}(\sqrt{5})$	419	20	$\mathbb{Q}(\sqrt{11})$
71	8	$\mathbb{Q}(\sqrt{15})$	209	14	$\mathbb{Q}(\sqrt{3})$	461	21	$\mathbb{Q}(\sqrt{437})$
	15	$\mathbb{Q}(\sqrt{3})$	239	15	$\mathbb{Q}(\sqrt{221})$	505	22	$\mathbb{Q}(\sqrt{30})$
76	21	$\mathbb{Q}(\sqrt{5})$		35	$\mathbb{Q}(\sqrt{2})$	521	144	$\mathbb{Q}(\sqrt{5})$
89	9	$\mathbb{Q}(\sqrt{77})$	265	56	$\mathbb{Q}(\sqrt{3})$	551	23	$\mathbb{Q}(\sqrt{21})$

Table from draft of [Appleby, Flammia, and K 2024+]. Some rows are aspirational; I've verified the ghost r-SIC construction works for $d \leq 100$.

Thank you!

Thank you for listening! Any questions?