Class field theory for nonmaximal orders (abstractly and explicitly)

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Abelian extensions of $\mathbb{Q}$

Question

Fix K a number field. What are the abelian Galois extensions of K?

Theorem (Kronecker–Weber)

For $K = \mathbb{Q}$, every abelian Galois extension is contained in some $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of 1.

Example (quadratic extensions of $\mathbb{Q}$)

$$\sqrt{2} = \zeta_8 + \zeta_8^{-1}, \qquad \text{so } \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8).$$

$$\sqrt{-7} = 1 + 2\zeta_7 + 2\zeta_7^2 + 2\zeta_7^4, \qquad \text{so } \mathbb{Q}(\sqrt{-7}) \subset \mathbb{Q}(\zeta_7).$$

Example (non-abelian extension of $\mathbb{Q}$)

For all $n \in \mathbb{N}$, $\sqrt[3]{2} \notin \mathbb{Q}(\zeta_n)$.

Abelian extensions of an imaginary quadratic field, unramified case

Fix K an imaginary quadratic field of discriminant Δ.
Then $\mathcal{O}_K = \mathbb{Z}[\omega] = \mathbb{Z} + \omega\mathbb{Z}$ where $\omega = \frac{\Delta+\sqrt{\Delta}}{2}$.

Theorem (Weber, Hilbert)

Let K^{ur} be the maximal unramified abelian extension of K. Then K^{ur}/K is a finite extension of degree $h_K = |\mathrm{Cl}(K)|$, it is the minimal extension such that there exists an elliptic curve

$$E = \{y^2 = x^3 + ax + b\} \text{ with } a, b \in K^{\mathrm{ur}} \text{ and } E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z}).$$

Example ($\mathbb{Q}(\frac{1+\sqrt{-7}}{2})^{\mathrm{ur}} = \mathbb{Q}(\frac{1+\sqrt{-7}}{2})$)

$E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} + \frac{1+\sqrt{-7}}{2}\mathbb{Z})$ for $E = \{y^2 = x^3 - 140x - 784\}$.

Example ($\mathbb{Q}(\sqrt{-6})^{\mathrm{ur}} = \mathbb{Q}(\sqrt{-3}, \sqrt{2})$)

$E(\mathbb{C}) \simeq \frac{\mathbb{C}}{\mathbb{Z}+\sqrt{-6}\mathbb{Z}}$ for $E = \{y^2 = x^3 - 3(95 + 4\sqrt{2})x + 46(7 + 17\sqrt{2})\}$.

Abelian extensions of an imaginary quadratic field, general case

Fix K an imaginary quadratic field of discriminant Δ.
Then $\mathcal{O}_K = \mathbb{Z}[\omega] = \mathbb{Z} + \omega\mathbb{Z}$ where $\omega = \frac{\Delta+\sqrt{\Delta}}{2}$.

Theorem (Weber, others)

Consider the complex torus $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$. Let H_m be the minimum field extension of K such that there exists an elliptic curve

$$E \text{ with } m\text{-torsion points in } H_m \text{ and } E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z}).$$

Then any abelian extension of K is contained in some H_m.

Observation (Weber, Söhngen)

The above theorem remains true if $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$ is replaced with $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for any $\tau \in K$. However, the H_m are replaced by larger fields $H_m^{\mathcal{O}}$ depending on the multiplier ring

$$\mathcal{O} = \mathrm{ord}(\mathbb{Z}+\tau\mathbb{Z}) = \{\alpha \in K : \alpha(\mathbb{Z}+\tau\mathbb{Z}) \subseteq \mathbb{Z}+\tau\mathbb{Z}\}.$$

Abelian extensions of an arbitrary number field

Class field theory (early 20th century work of Artin, Furtwängler, Hasse, Hilbert, Takagi, Weber, ...) establishes, for any number field K:

- Ray class groups $\mathrm{Cl}_{\mathfrak{m},\Sigma}(K)$ defined as refined quotients of ideal groups.
- Ray class fields $H_{\mathfrak{m},\Sigma}$ having the properties:
  - $\mathrm{Gal}(H_{\mathfrak{m},\Sigma}/K) \cong \mathrm{Cl}_{\mathfrak{m},\Sigma}(K)$.
  - $H_{\mathfrak{m},\Sigma}$ may be specified using prime splitting conditions.
  - Every abelian extension of K is contained in some $H_{\mathfrak{m},\Sigma}$.

  These generalize the fields H_m in the imaginary quadratic case.

Remark

The ray class fields are not defined "explicitly" and are difficult to calculate computationally.

Class field theory for orders of an arbitrary number field

This talk (work of K and Lagarias) will present an approach to class field theory based on more general class groups and class fields:

- Ray class groups of orders $\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$ defined as refined quotients of invertible ideal groups.
- Ray class fields $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$ having the properties:
  - $\mathrm{Gal}(H^{\mathcal{O}}_{\mathfrak{m},\Sigma}/K) \cong \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$.
  - $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$ may be specified using splitting conditions of prime ideals of $\mathcal{O}$.
  - Every abelian extension of K is contained in some $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$.

These generalize the fields $H^{\mathcal{O}}_{m}$ in the imaginary quadratic case. The approach complements an idélic approach to “ray class fields for orders” by independent work of Campagna and Pengo (2021).

Orders of number fields and fractional ideals

Definition

Let K be a number field. An order $\mathcal{O} \subset K$ is a subring for which

- $\mathcal{O}$ has finite rank as a $\mathbb{Z}$-module. (Excludes, e.g., $\mathbb{Z}[\frac{1}{2}]$ in $\mathbb{Q}$.)
- $\mathbb{Q}\mathcal{O} = K$. (Excludes, e.g., $\mathbb{Z}$ in $\mathbb{Q}(i)$.)

The ring of integers $\mathcal{O}_K$ is the maximal order. Examples of nonmaximal orders include $\mathbb{Z}[\sqrt{5}]$ in $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Z}[3i]$ in $\mathbb{Q}(i)$.

Definition

A fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is a finite rank $\mathcal{O}$-submodule of K. (Equivalently, some integer multiple $n\mathfrak{a}$ is in integral ideal of $\mathcal{O}$.)

Any lattice $\Lambda \subset K$ is a fractional ideal of its multiplier ring

$$\operatorname{ord}(\Lambda) = \{\alpha \in K : \alpha\Lambda \subseteq \Lambda\}.$$

Invertible fractional ideals

Let $\mathcal{O}$ be an order in a number field K.

Definition

The fractional ideal $\mathfrak{a}$ is invertible if there exists another fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}$.

Example

$2\mathbb{Z} + 2i\mathbb{Z}$ is **not** invertible as a $\mathbb{Z}[2i]$-ideal but **is** invertible as a $\mathbb{Z}[i]$-ideal.

Example

$\mathfrak{q} = 2\mathbb{Z} + 2\sqrt[3]{2}\mathbb{Z} + 4\sqrt[3]{4}$ is an ideal of $\mathrm{ord}(\mathfrak{q}) = \mathbb{Z} + 2\sqrt[3]{2}\mathbb{Z} + 2\sqrt[3]{4}$, but it is **not** an invertible ideal of **any** order.

Colons, conductors, and coprimality

Definition

For any two fractional ideals $\mathfrak{a}$, $\mathfrak{b}$ of orders in K, the colon ideal is

$$(\mathfrak{a} : \mathfrak{b}) = \{\alpha \in K : \alpha\mathfrak{b} \subseteq \mathfrak{a}\}.$$

Special cases are the conductor ideal of an order $\mathfrak{f}(\mathcal{O}) = (\mathcal{O} : \mathcal{O}_K)$ and the multiplier ring of an ideal $\mathrm{ord}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$.

Definition

A fractional $\mathcal{O}$-ideal $\mathfrak{c}$ is coprime to an integral $\mathcal{O}$-ideal $\mathfrak{m}$ if $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ where $\mathfrak{b}$ is invertible, $\mathfrak{a} + \mathfrak{m} = \mathcal{O}$, and $\mathfrak{b} + \mathfrak{m} = \mathcal{O}$.

Example

The fractional ideal $\mathfrak{c} = i\mathbb{Z}[2i]$ is **not** coprime to the conductor $\mathfrak{f}(\mathbb{Z}[2i]) = 2\mathbb{Z}[i]$ even though $\mathfrak{c}^2 = \mathbb{Z}[2i]$.

Proposition

Fractional $\mathcal{O}$-ideals coprime to $\mathfrak{f}(\mathcal{O})$ are invertible.

Ray class groups of orders

Let $\mathcal{O}$ be an order in a number field K. Fix

- $\mathfrak{m}$ an ideal of $\mathcal{O}$;
- Σ a subset of the real embeddings $\{\rho : K \to \mathbb{R}\}$.

Definition (K and Lagarias 2022)

The ray class group is defined as a quotient group

$$\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) = \frac{\mathrm{J}^*_{\mathfrak{m}}(\mathcal{O})}{\mathrm{P}_{\mathfrak{m},\Sigma}(\mathcal{O})}$$

where

$$\mathrm{J}^*_{\mathfrak{m}}(\mathcal{O}) = \{\text{invertible fractional ideals of } \mathcal{O} \text{ coprime to } \mathfrak{m}\}, \text{ and}$$
$$\mathrm{P}_{\mathfrak{m},\Sigma}(\mathcal{O}) = \{\alpha\mathcal{O} : \alpha \in K^\times, \alpha \equiv 1 \pmod{\mathfrak{m}}, \rho(\alpha) > 0 \text{ for all } \rho \in \Sigma\}.$$

Change of modulus

We need to add further coprimality conditions to the definition of the ray class group to allow us to

- define maps between different ray class groups for the same order, and
- make the invertibility condition follow from the coprimality condition.

Proposition (K and Lagarias 2022)

For any $\mathcal{O}$-ideal $\mathfrak{d} \subseteq \mathfrak{m}$, the inclusion map $\mathrm{J}^*_{\mathfrak{d}}(\mathcal{O}) \subseteq \mathrm{J}^*_{\mathfrak{m}}(\mathcal{O})$ induces an isomorphism

$$\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) \cong \frac{\mathrm{J}^*_{\mathfrak{d}}(\mathcal{O})}{\mathrm{P}^{\mathfrak{d}}_{\mathfrak{m},\Sigma}(\mathcal{O})}$$

where

$$\mathrm{J}^*_{\mathfrak{d}}(\mathcal{O}) = \{\text{invertible fractional ideals of } \mathcal{O} \text{ coprime to } \mathfrak{d}\}, \text{ and}$$

$$\mathrm{P}^{\mathfrak{d}}_{\mathfrak{m},\Sigma}(\mathcal{O}) = \left\{ \alpha\mathcal{O} : \begin{array}{l} \alpha \in K^\times, \alpha \equiv 1 \pmod{\mathfrak{m}}, \rho(\alpha) > 0 \text{ for all } \rho \in \Sigma, \\ \text{and } \alpha\mathcal{O} \text{ is coprime to } \mathfrak{d} \end{array} \right\}.$$

Change of order

We also need to define maps between class groups for different orders. For $\mathcal{O} \subseteq \mathcal{O}'$, there are extension and contraction maps

$$\begin{aligned} \text{ext} : \{\mathcal{O}\text{-ideals}\} &\to \{\mathcal{O}'\text{-ideals}\} & \text{ext}(\mathfrak{a}) &= \mathfrak{a}\mathcal{O}, \\ \text{con} : \{\mathcal{O}'\text{-ideals}\} &\to \{\mathcal{O}\text{-ideals}\} & \text{con}(\mathfrak{a}') &= \mathfrak{a} \cap \mathcal{O}. \end{aligned}$$

The main technical result about extension and contraction maps is the following proposition.

Proposition

Let $\mathfrak{d}$ be an integral $\mathcal{O}'$-ideal with $\mathfrak{d} \subseteq (\mathcal{O} : \mathcal{O}')$. Then the extension and contraction maps extend uniquely to isomorphisms

$$\begin{aligned} \text{ext} &: J_{\mathfrak{d}}(\mathcal{O}) \to J_{\mathfrak{d}}(\mathcal{O}'), \\ \text{con} &: J_{\mathfrak{d}}(\mathcal{O}') \to J_{\mathfrak{d}}(\mathcal{O}). \end{aligned}$$

This is technically tricky because con is **not** a homomorphism on ideals not coprime to $(\mathcal{O} : \mathcal{O}')$.

Change of modulus and order exact sequence

Theorem (K and Lagarias 2022)

Let $\mathfrak{m}$ be an ideal of $\mathcal{O}$, $\mathfrak{m}'$ an ideal of $\mathcal{O}'$ such that $\mathfrak{m}\mathcal{O}' \subseteq \mathfrak{m}'$, and $\Sigma' \subseteq \Sigma \subseteq \{\text{embeddings } K \hookrightarrow \mathbb{R}\}$. Let $\mathfrak{d}$ be any $\mathcal{O}'$-ideal such that $\mathfrak{d} \subseteq (\mathfrak{m} : \mathcal{O}')$. Let $r = |\Sigma \setminus \Sigma'|$. We have the following exact sequence.

$$1 \to \frac{\mathrm{U}_{\mathfrak{m}',\Sigma'}(\mathcal{O}')}{\mathrm{U}_{\mathfrak{m},\Sigma}(\mathcal{O})} \to \frac{\mathrm{U}_{\mathfrak{m}'}(\mathcal{O}'/\mathfrak{d})}{\mathrm{U}_{\mathfrak{m}}(\mathcal{O}/\mathfrak{d})} \times \{\pm 1\}^r \to \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) \to \mathrm{Cl}_{\mathfrak{m}',\Sigma'}(\mathcal{O}') \to 1.$$

Here, the “U-groups” are subgroups on unit groups defined by

$$\mathrm{U}_{I,\Sigma}(R) := \{\alpha \in R^\times : \alpha \equiv 1 \pmod{I} \text{ and } \rho(\alpha) > 0 \text{ for } \rho \in \Sigma\}.$$

This theorem allows us to

- understand the kernel of the “extension and change of modulus map” between different ray class groups of orders;
- construct new maps (needed for class field construction);
- compute the sizes of ray class groups of orders.

Construction of class field of orders

A surjective map from $\psi : \mathrm{Cl}_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}(\mathcal{O}_K) \to \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$ is induced by the following diagram (and may also be described more explicitly using the contraction map).

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \dfrac{\mathcal{O}_K^\times}{\mathrm{U}_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}(\mathcal{O}_K)} & \longrightarrow & (\mathcal{O}_K/(\mathfrak{m}:\mathcal{O}_K))^\times \times \{\pm 1\}^{|\Sigma|} & \longrightarrow & \mathrm{Cl}_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}(\mathcal{O}_K) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_K) & \longrightarrow & 1 \\
 & & \Big\downarrow \kappa & & \Big\downarrow \pi & & \Big\downarrow \psi & & \Big\| \mathrm{id} & & \\
1 & \longrightarrow & \dfrac{\mathcal{O}_K^\times}{\mathrm{U}_{\mathfrak{m},\Sigma}(\mathcal{O})} & \longrightarrow & \dfrac{(\mathcal{O}_K/(\mathfrak{m}:\mathcal{O}_K))^\times}{\mathrm{U}_{\mathfrak{m}}(\mathcal{O}/(\mathfrak{m}:\mathcal{O}_K))} \times \{\pm 1\}^{|\Sigma|} & \longrightarrow & \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_K) & \longrightarrow & 1
\end{array}$$

Then the Takagi existence theorem and Artin reciprocity (in standard class field theory) and the Galois correspondence defines a corresponding class field $H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$ as a subfield of $H_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}$.

Theorem 1

The following theorem describes the ray class field of an order in terms of the splitting of primes.

Theorem (K and Lagarias 2022)

The field $H_{\mathfrak{m},\Sigma}^{\mathcal{O}}$ is the unique abelian Galois extension of K with the property that:

a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ that is coprime to $(\mathfrak{m} : \mathcal{O}_K)$ splits completely in $H_{\mathfrak{m},\Sigma}^{\mathcal{O}}$ $\iff$ $\mathfrak{p} \cap \mathcal{O} = \pi\mathcal{O}$, a principal prime $\mathcal{O}$-ideal such that $\pi \equiv 1 \pmod{\mathfrak{m}}$ and $\rho(\pi) > 0$ for $\rho \in \Sigma$.

Theorem 2

The following theorem relates the ray class field of an order to ray class fields of a larger order (such as the maximal order).

Theorem (K and Lagarias 2022)

For any order $\mathcal{O}' \supseteq \mathcal{O}$, there are inclusions of ray class fields

$$H^{\mathcal{O}'}_{\mathfrak{m}\mathcal{O}',\Sigma} \subseteq H^{\mathcal{O}}_{\mathfrak{m},\Sigma} \subseteq H^{\mathcal{O}'}_{(\mathfrak{m}:\mathcal{O}'),\Sigma}.$$

In particular, for $\mathcal{O}' = \mathcal{O}_K$,

$$H_{\mathfrak{m}\mathcal{O}_K,\Sigma} \subseteq H^{\mathcal{O}}_{\mathfrak{m},\Sigma} \subseteq H_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}.$$

Theorem 3

The following theorem is a generalization of the Artin reciprocity law to ray class fields of orders.

Theorem (K and Lagarias 2022)

Let $H_0 = H^{\mathcal{O}}_{\mathfrak{m},\Sigma}$ and $H_1 = H_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}$. There is an isomorphism

$$\mathrm{Art}_{\mathcal{O}} : \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) \to \mathrm{Gal}(H_0/K),$$

uniquely determined by the condition on prime ideals

$$\mathrm{Art}_{\mathcal{O}}([\mathfrak{p}])(\alpha) \equiv \alpha^p \pmod{\mathfrak{P}}$$

where $\mathfrak{P}$ is any prime of $\mathcal{O}_{H_0}$ lying over $\mathfrak{p}\mathcal{O}_K$. Moreover, for any ideal $\mathfrak{a}$ coprime to $\mathfrak{f}(\mathcal{O}) \cap \mathfrak{m}$,

$$\mathrm{Art}_{\mathcal{O}}([\mathfrak{a}]) = \mathrm{Art}([\mathfrak{a}\mathcal{O}_K])|_{H_0},$$

where $\mathrm{Art} : \mathrm{Cl}_{(\mathfrak{m}:\mathcal{O}_K),\Sigma}(\mathcal{O}_K) \to \mathrm{Gal}(H_1/K)$ is the usual Artin map of class field theory.

Explicit class field theory, overview

Hilbert's 12th Problem asks for analytic functions whose special values generate the abelian extensions of a number field F.

We really want:

(1) analytic functions

(2) a geometric explanation

Stark's conjecture provides a partial answer to (1) via L-functions. We still don't know much about (2).

I won't talk about:

- CM abelian varieties in $\dim > 1$
- p-adic solutions (Gross–Stark and Brumer–Stark conjectures; work of Dasgupta, Kakde, Silliman, Ventullo, Wang; work of Darmon, Pozzi, Vonk)

Explicit class field theory, analytic approach

field F	analytic function values generating $H^{\mathcal{O}}_{m\mathcal{O},\Sigma}$
$\mathbb{Q}$	$\exp\left(\frac{2\pi i}{m}\right)$
imaginary quadratic	complex multiplication values of modular functions (e.g., the Klein j-function and Weber's functions/modular units) of level m
real quadratic	conjecturally, stable (RM) values $\mathfrak{w}^{\mathfrak{r}}[\beta]$ of the level m Shintani–Faddeev modular cocycle [K 2024+; see also Stark 1976, Shintani 1977, Kurokawa 1991, Sczech 1993, ...]
complex cubic	conjecturally, at least for $\mathcal{O} = \mathcal{O}_F$, stable values of a cocycle related to the elliptic gamma function [Bergeron, Charollois, García 2023]

Spotlight: The Shintani–Faddeev modular cocycle

Let $\mathbf{r} = \left(\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right) \in \mathbb{Q}^2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathbf{r}} = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A\mathbf{r} - \mathbf{r} \in \mathbb{Z}^2\}$.
For $\mathrm{Im}(\tau) > 0$, define

$$\varpi_A^{\mathbf{r}}(\tau) = \prod_{k=0}^{\infty} \frac{1 - \mathrm{e}\left((k + r_2)\frac{a\tau+b}{c\tau+d} - r_1\right)}{1 - \mathrm{e}((k + r_2)\tau - r_1)}, \text{ where } \mathrm{e}(z) := e^{2\pi i z}.$$

Theorem (Dimofte 2015; K 2024+)

The function $\varpi_A^{\mathbf{r}}(\tau)$ meromorphically continues (with some poles in $\mathbb{Q}$) to $U_A = \mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\}$.

Theorem (K 2024+)

Suppose β is real quadratic and $\frac{a\beta+b}{c\beta+d} = \beta$. Assuming the Stark conjectures, if $\mathcal{O} = \mathrm{ord}(\beta\mathbb{Z} + \mathbb{Z})$ and $\mathbf{r} \in \frac{1}{m}\mathbb{Z}^2$, then

$$(\text{explicit } 12m\text{-th root of unity})(\varpi_A^{\mathbf{r}}(\beta))^2 \in H^{\mathcal{O}}_{m\mathcal{O},\{\infty_2\}}.$$

Explicit class field theory, algebraic/geometric approach

field F	geometric object producing $H^{\mathcal{O}}_{m\mathcal{O},\Sigma}$
$\mathbb{Q}$	m-torsion points on the unit circle
imaginary quadratic	m-torsion points on elliptic curve E with $E(\mathbb{C}) \cong \mathbb{C}/\mathcal{O}$
real quadratic	conjecturally (for a restricted set of m), certain equichordal configurations of m^2 subspaces in $\mathbb{C}^m$ [Appleby, Flammia, K 2024+]
complex cubic	???

Spotlight: Sample of Appleby–Flammia–K construction

Let $F = \mathbb{Q}(\sqrt{5})$ and $\varphi = \frac{1+\sqrt{5}}{2}$.

- $H^{\mathbb{Z}[\varphi]}_{(4)\infty_1\infty_2} = H^{\mathbb{Z}[\varphi]}_{4\mathbb{Z}[\varphi],\{\infty_1,\infty_2\}}$ is generated by the unitary invariants of 16 equaiangular lines in $\mathbb{C}^4$.
- $H^{\mathbb{Z}[\varphi]}_{(8)\infty_1\infty_2}$ and $H^{\mathbb{Z}[2\varphi]}_{(8)\infty_1\infty_2}$ are generated by the unitary invariants of two different configurations of 64 equaiangular lines in $\mathbb{C}^8$.
- $H^{\mathbb{Z}[\varphi]}_{(11)\infty_1\infty_2}$ is generated by the unitary invariants of 121 equichordal 3-dimensional subspaces in $\mathbb{C}^{11}$.
- Assuming the Stark conjectures and a (probably very difficult!) special value identity for $\mathfrak{w}^{\mathfrak{r}}_A(\tau)$, a cofinal set of class fields are obtained similarly.

Other potential applications

- Complex multiplication of abelian varieties (non-Gorenstein orders cause difficulties; talk to Pete Clark)
- (Higher) composition laws (and arithmetic statistics)
- Modular-eque q-series $\sum_{\mathcal{O}} c_{\mathcal{O}} q^{\pm \mathrm{disc}(\mathcal{O})}$ built from ray class data over some family of orders (Beckwith and K work in progress)
- Lattice-based cryptography (some existing schemes use non-maximal orders)
- Computational algebraic number theory

Thank you!

Thank you for listening! Any questions?