The Shintani–Faddeev modular cocycle: Stark units from q-Pochhammer ratios

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L-functions at $s = 1$: example with base field $\mathbb{Q}$

The following formula can be proved using calculus. Try it!

Example (Exercise)

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

- The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.
- On the right-hand side is $1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.
- This talk concerns higher analogs of this formula in the framework of the Stark conjectures.

Ray class groups and ray class fields

Let F be a number field, $\mathfrak{m}$ a nonzero ideal of $\mathcal{O}_F$, and Σ be a subset of the real embeddings of F.

Definition (Weber–Takagi–Hasse 1897–1926)

The ray class group modulo $(\mathfrak{m}, \Sigma)$ is

$$\mathrm{Cl}_{\mathfrak{m},\Sigma} = \frac{\{\text{fractional ideals of } \mathcal{O}_F \text{ coprime to } \mathfrak{m}\}}{\{a\mathcal{O} \text{ s.t. } a \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}}.$$

Class field theory associates to $\mathrm{Cl}_{\mathfrak{m},\Sigma}$ a ray class field $H_{\mathfrak{m},\Sigma}$, an abelian extension of F with Galois group $\mathrm{Gal}(H_{\mathfrak{m},\Sigma}/F) = \mathrm{Cl}_{\mathfrak{m},\Sigma}$.

Varying $\mathfrak{m}$ and Σ, the ray class fields are cofinal among all abelian extensions of F.

Zeta functions associated to ray classes

Definition

For $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}$, the partial zeta function is

$$\zeta_{\mathfrak{m},\Sigma}(s,\mathfrak{A}) = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O} \\ \mathfrak{a}\in\mathfrak{A}}} \mathrm{Nm}(\mathfrak{a})^{-s}.$$

Let $\mathfrak{R} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}$ be the ray ideal class

$$\mathfrak{R} = \{a\mathcal{O} : a \equiv -1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}.$$

Definition

For $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}$, the differenced partial zeta function is

$$Z_{\mathfrak{m},\Sigma}(s,\mathfrak{A}) = \zeta_{\mathfrak{m},\Sigma}(s,\mathfrak{A}) - \zeta_{\mathfrak{m},\Sigma}(s,\mathfrak{R}\mathfrak{A}).$$

Eta and theta functions

The Dedekind eta function is

$$\eta(\tau) = \mathrm{e}\left(\frac{\tau}{24}\right) \prod_{k=1}^{\infty}(1 - \mathrm{e}(k\tau)), \text{ where } \mathrm{e}(z) = e^{2\pi i z} \text{ and } \tau \in \mathbb{H}.$$

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $A \cdot \tau = \frac{a\tau+b}{c\tau+d}$, it satisfies the modular transformation law

$$\eta(A \cdot \tau) = \psi\left(A, \sqrt{c\tau + d}\right) \sqrt{c\tau + d}\, \eta(\tau).$$

The eta-multiplier character $\psi : \mathrm{Mp}_2(\mathbb{Z}) \to \langle \zeta_{24} \rangle$ can be described explicitly in terms of Jacobi symbols or Dedekind sums.

Eta and theta functions

The Jacobi theta function with characteristics $\mathbf{r} = \left(\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right) \in \mathbb{Q}^2$ is

$$\theta_{\mathbf{r}}(\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}\big(\tfrac{1}{2}(n + r_2 + \tfrac{1}{2})^2\tau + (n + r_2 + \tfrac{1}{2})(-r_1 + \tfrac{1}{2})\big)\,.$$

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathbf{r}}$ where

$$\Gamma_{\mathbf{r}} = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A\mathbf{r} - \mathbf{r} \in \mathbb{Z}^2\},$$

the theta function satisfies the modular transformation law

$$\theta_{\mathbf{r}}(A\cdot\tau) = \psi\Big(A, \sqrt{c\tau + d}\Big)^3 \chi_{\mathbf{r}}(A)\sqrt{c\tau + d}\,\theta_{\mathbf{r}}(\tau)$$

for a particular character $\chi_{\mathbf{r}} : \mathrm{SL}_2(\mathbb{Z}) \to \langle\zeta_{2N}\rangle$, with $N = \mathrm{LCD}[r_1, r_2]$.

Kronecker second limit formula (applied to imaginary quadratic fields)

Theorem (Kronecker)

- Let F be an imaginary quadratic number field.
- Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be a nonzero ideal with $\mathfrak{m} \neq \mathcal{O}_F$.
- Let $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m},\emptyset}$ and $\mathfrak{A}_0$ the class of $\mathfrak{A}$ in $\mathrm{Cl}_{\mathcal{O}_F,\emptyset}$.
- Choose $\mathfrak{b} \in \mathfrak{A}_0^{-1}$ coprime to $\mathfrak{m}$.
- Write $\mathfrak{b}\mathfrak{m} = \alpha(\tau\mathbb{Z} + \mathbb{Z})$ for $\tau \in \mathbb{H}$.
- Choose $\mathbf{r} = \binom{r_1}{r_2} \in \mathbb{Q}^2$ such that $(\alpha(r_2\tau - r_1))\mathfrak{b}^{-1} \in \mathfrak{A}$.

Then, for $W = \left|\mathcal{O}_F^\times\right|$ (so $W = 2$ for $F \neq \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1})$),

$$\exp\left(-\zeta'_{\mathfrak{m},\emptyset}(0,\mathfrak{A})\right) = \left|\frac{\theta_{\mathbf{r}}(\tau)}{\eta(\tau)}\right|^W.$$

Moreover, an integer power of the "CM value" $\frac{\theta_{\mathbf{r}}(\tau)}{\eta(\tau)}$ is an "elliptic unit" in the ray class field $H_{\mathfrak{m},\emptyset}$ over F.

Shintani limit formula and Stark conjecture

Theorem (Shintani 1977)

- Let F be an real quadratic number field.
- Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be a nonzero ideal with $\mathfrak{m} \neq \mathcal{O}_F$.
- Let $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m},\{\infty_2\}}$.

Then,

$$\exp\left(-Z'_{\mathfrak{m},\{\infty_2\}}(0,\mathfrak{A})\right) = X(\mathfrak{A})$$

where $X(\mathfrak{A})$ is an explicit finite (but arbitrarily long) product of special values of the Barnes–Shintani double sine function.

Stark's original conjecture in the rank 1 abelian real quadratic case (1976) says that $\exp\left(-Z'_{\mathfrak{m},\{\infty_2\}}(0,\mathfrak{A})\right)$ is an algebraic unit in $H_{\mathfrak{m},\{\infty_2\}}$.

Tate's refinement (1981) includes the statement that $\exp\left(-\frac{1}{2}Z'_{\mathfrak{m},\{\infty_2\}}(0,\mathfrak{A})\right)$ is in an abelian extension of F.

Goals for this talk

We write the invariant $X(\mathfrak{A})$ appearing in Shintani's formula as:

- a limit from the upper half plane of a relatively simple infinite product...
- ...which can be interpreted as a real multiplication value of a cocycle on $\Gamma_{\mathbf{r}}$ valued in complex meromorphic functions.

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i \tau}$, set $\varpi_{\binom{0}{4/5}}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

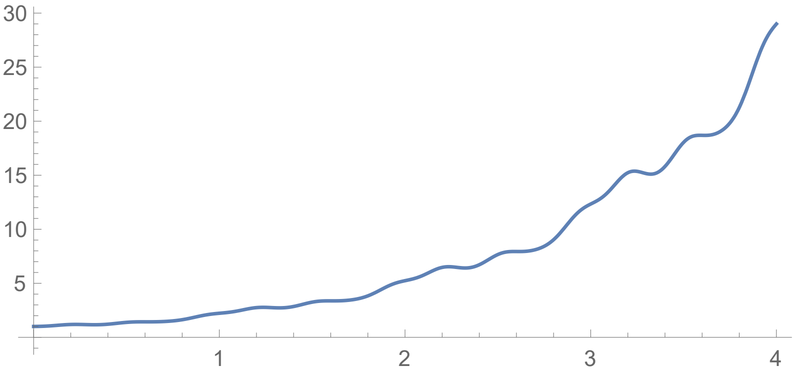

Figure: Graph of $y = \left|\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$

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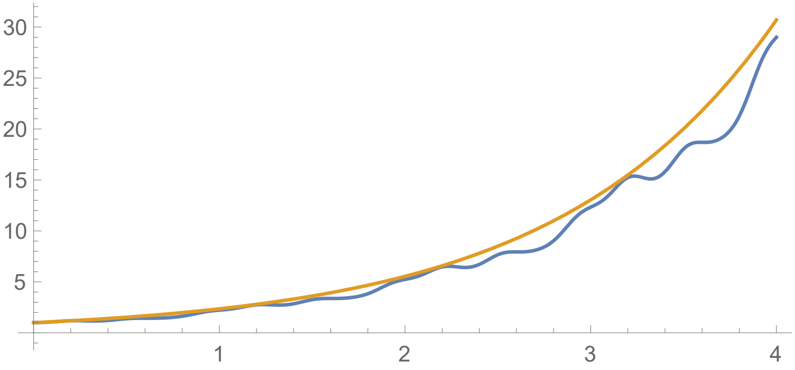

Figure: Graphs of $y = \left|\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$ and $y = (2.35385)^t$

q-Pochhammer asymptotics near real quadratic points: example

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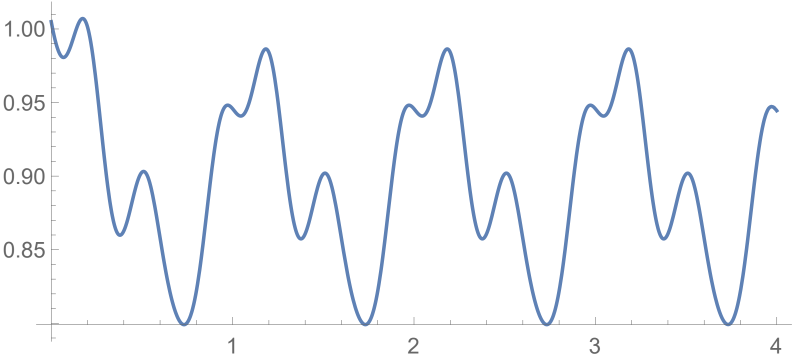

Figure: Graph of $y = \left|\nu^{-t}\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$; $\nu \approx 2.35385e^{-\frac{7\pi i}{20}}$.

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i \tau}$, set $\varpi_{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

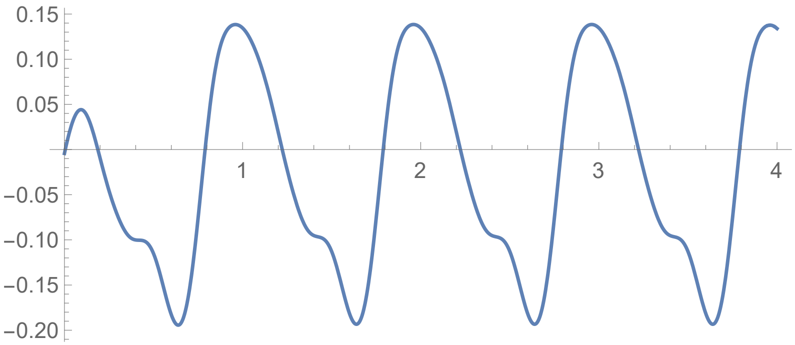

Figure: Graph of $y = \arg\left(\nu^{-t}\varpi_{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right)$.

Interpreting the base of the exponential

What is the meaning of the number $\nu \approx 2.35385 e^{-\frac{7\pi i}{20}}$?

- The phase $e^{-\frac{7\pi i}{20}}$ can be obtained up to ± 1 from the modularity of $(q^{1/5}, q)_\infty (q^{4/5}, q)_\infty$.
- Our main theorem implies that

$$|\nu|^{-1} = \exp(-\tfrac{1}{2} Z'_{\mathfrak{m},\{\infty_2\}}(0, \mathfrak{A})) = \exp(-\zeta'_{\mathfrak{m},\{\infty_2\}}(0, \mathfrak{A}))$$

  for a certain ray ideal class $\mathfrak{A} \in \mathrm{Cl}_{(5),\{\infty_2\}}$.
- Assuming the Stark conjectures, $|\nu|^{-2}$ is an algebraic unit in an abelian Galois extension of $\mathbb{Q}(\sqrt{3})$ (a “Stark unit”).
- Tate’s refinement implies that $\mathbb{Q}(\nu)$ is also abelian over $\mathbb{Q}(\sqrt{3})$.

Stark unit example

The number $|\nu|^2 \approx 5.54061$ appears to be a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

This root is an algebraic unit in the ray class field $H = H_{(5),\{\infty_2\}}$, with $\mathrm{Gal}\Big(H/\mathbb{Q}(\sqrt{3})\Big) \cong \mathbb{Z}/8\mathbb{Z}$.

The number $|\nu| \approx 2.35385$ appears to be an element of $H(\sqrt{2})$.

The variant Pochhammer function with characteristics

For $\mathbf{r} \in \left(\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right) \in \mathbb{Q}^2$, $\tau \in \mathbb{H}$, and $\mathrm{e}(z) := e^{2\pi i z}$, consider the function

$$\varpi_{\mathbf{r}}(\tau) = (\mathrm{e}(r_2\tau - r_1), \mathrm{e}(\tau))_\infty = \prod_{k=0}^{\infty} (1 - \mathrm{e}((k + r_2)\tau - r_1)).$$

Question

How does $\varpi_{\mathbf{r}}(\tau)$ transform under fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$?

Modularity?

$$\varpi_{\mathbf{r}}(\tau) = (\mathrm{e}(r_2\tau - r_1), \mathrm{e}(\tau))_\infty = \prod_{k=0}^{\infty} (1 - \mathrm{e}((k + r_2)\tau - r_1)).$$

In particular, taking $q = \mathrm{e}(\tau)$:

$$\begin{aligned}
\varpi_{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}(\tau) &= (q,q)_\infty &&= q^{-1/24}\eta(\tau);\\
\varpi_{\left(\begin{smallmatrix}0\\1/2\end{smallmatrix}\right)}(\tau) &= \frac{(q^{1/2}, q^{1/2})_\infty}{(q,q)_\infty} &&= q^{1/48}\frac{\eta(\tau/2)}{\eta(\tau)};\\
\varpi_{\left(\begin{smallmatrix}1/2\\1\end{smallmatrix}\right)}(\tau) &= \frac{(q^2, q^2)_\infty}{(q,q)_\infty} &&= q^{-1/24}\frac{\eta(2\tau)}{\eta(\tau)};\\
\varpi_{\left(\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right)}(\tau) &= \frac{(-q^{1/2}, -q^{1/2})_\infty}{(q,q)_\infty} &&= \zeta_{48} q^{1/48}\frac{\eta((\tau+1)/2)}{\eta(\tau)}.
\end{aligned}$$

Up to polynomials in fractional powers of q,

$$\varpi_{\mathbf{r}}(\tau) \text{ is weakly modular } \iff \mathbf{r} \in \tfrac{1}{2}\mathbb{Z}^2.$$

Connection to theta functions

Question: What happens for $\mathbf{r} \notin \frac{1}{2}\mathbb{Z}^2$?

Standard answer: We're missing half the product to get a theta function.

Theorem (Jacobi, rephrased)

The variant Pochhammer function with characteristics is related to the theta function with characteristics by

$$\varpi_{\mathbf{r}}(\tau)\varpi_{-\mathbf{r}}(\tau) = (*)\frac{\theta_{\mathbf{r}}(\tau)}{\eta(\tau)}$$

$$(*) = i\,\mathrm{e}\!\left(-\left(\frac{r_2^2}{2}+\frac{1}{12}\right)\tau - r_2\left(-r_1+\frac{1}{2}\right)\right)\left(\mathrm{e}\!\left(\frac{r_2\tau-r_1}{2}\right)-\mathrm{e}\!\left(\frac{-r_2\tau+r_1}{2}\right)\right)$$

Proof sketch: This formula follows after a change of variables from the Jacobi triple product identity.

Modular transformations of the variant Pochhammer function

Question: What happens for $\mathbf{r} \notin \frac{1}{2}\mathbb{Z}^2$?

Alternative answer: We can weaken our notion of modularity.

Theorem (as stated by K 2024)

For each $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathbf{r}}$, there is a meromorphic function $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ on $\mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\}$ with the property that

$$\varpi_{\mathbf{r}}(A \cdot \tau) = \mathfrak{w}^{\mathbf{r}}_A(\tau)\varpi_{\mathbf{r}}(\tau).$$

Remark: Ideas and results of Shintani (1977), Arakawa (1982), Faddeev (1994), Yamamoto (2010), Dimofte (2015), Sarkissian and Spiridonov (2020), and Garoufalidis and Wheeler (2022) are closely related to this theorem.

Modular transformations of the variant Pochhamer function, proof sketch

For $A = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\varpi_{T\mathbf{r}}(T \cdot \tau) = \varpi_{\mathbf{r}}(\tau)$.

For $A = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$...

Theorem (Shintani 1977, rephrased)

$$\left(\mathrm{e}\left(\frac{z}{\tau}\right), \mathrm{e}\left(-\frac{1}{\tau}\right)\right)_\infty = \mathrm{e}\left(\frac{\tau - 3 + \tau^{-1}}{24} + \frac{(\tau - z)(1-z)}{4\tau}\right)\left(1 - \mathrm{e}\left(\frac{z}{\tau}\right)\right) \cdot \mathrm{Sin}_2(z, \tau)\,(\mathrm{e}(z), \mathrm{e}(\tau))_\infty,$$

where $\mathrm{Sin}_2(z, \tau)$ is the double sine function, a meromorphic function of $z \in \mathbb{C}$ and $\tau \in \mathbb{C} \setminus (-\infty, 0]$.

Thus, $\varpi_{S\mathbf{r}}(S \cdot \tau) = s_S^{\mathbf{r}}(\tau)\varpi_{\mathbf{r}}(\tau)$ for a meromorphic function $s_S^{\mathbf{r}}(\tau)$ on $\mathbb{C} \setminus (-\infty, 0]$.

Modular transformations of the variant Pochhamer function, proof sketch

For general $A \in \mathrm{SL}_2(\mathbb{Z})$, decompose A as a product of S and T matrices.

Obtain a formula

$$\varpi_{A\mathbf{r}}(A \cdot \tau) = s_A^{\mathbf{r}}(\tau)\varpi_{\mathbf{r}}(\tau);$$

some care is required in choosing the decomposition to obtain $s_A^{\mathbf{r}}(\tau)$ meromorphic on $\mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\}$.

Specializing to $A \in \Gamma_{\mathbf{r}}$ yields the theorem.

The double sine function

Shintani's definition of $\mathrm{Sin}_2(z,\tau)$ is in terms of functions previously defined by Barnes. The **double zeta function** is

$$\zeta_2(s,z;\omega_1,\omega_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(z+\omega_1 m+\omega_2 n)^{-s}.$$

The **double gamma function** is

$$\Gamma_2(z;\omega_1,\omega_2)=\rho_2(\omega_1,\omega_2)\exp\left(\frac{d}{ds}\zeta_2(s,z;\omega_1,\omega_2)\Big|_{s=0}\right).$$

The **double sine function** is

$$\mathrm{Sin}_2(z;\omega_1,\omega_2)=\frac{\Gamma_2(\omega_1+\omega_2-z;\omega_1,\omega_2)}{\Gamma_2(z;\omega_1,\omega_2)}.$$

Because $\mathrm{Sin}_2(\alpha z;\alpha\omega_1,\alpha\omega_2)=\mathrm{Sin}_2(z;\omega_1,\omega_2)$, we lose no generality by defining $\mathrm{Sin}_2(z,\tau)=\mathrm{Sin}_2(z;\tau,1)$.

A working definition of a first cohomology group

- X a topological space, X° an open subspace
- $\mathcal{F}$ a sheaf of multiplicative groups of $\mathbb{C}$-valued functions on X
- Γ a discrete group acting continuously on X
- $U = (U_A)_{A\in\Gamma}$ a Γ-indexed open cover of X°

For $A \in \Gamma$ and $f \in \mathcal{F}(V)$, define $f^A \in \mathcal{F}(A^{-1} \cdot V)$ by $f^A(x) = f(A \cdot x)$. Set

$$C^1_U(\Gamma, \mathcal{F}) = \prod_{A\in\Gamma} \mathcal{F}(U_A);$$

$$Z^1_U(\Gamma, \mathcal{F}) = \{w \in C^1_U(\Gamma, \mathcal{F}) : w_{A_1A_2} = w_{A_1}^{A_2} w_{A_2}\};$$

$$B^1_U(\Gamma, \mathcal{F}) = \{w \in C^1_U(\Gamma, \mathcal{F}) : w_A = f^A f^{-1} \text{ for some } f \in \mathcal{F}(X^\circ)\};$$

$$H^1_U(\Gamma, \mathcal{F}) = \frac{Z^1_U(\Gamma, \mathcal{F})}{B^1_U(\Gamma, \mathcal{F})}.$$

Cohomological interpretation and examples

Now suppose $X = \mathbb{C} \cup \{\infty\}$, $X^\circ = \mathbb{C}$, $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, and $\mathcal{F} = \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}}$ the sheaf of nonzero meromorphic functions with poles restricted to $\mathbb{Q}$.

- For any Γ and $U_A = \mathbb{C}$, and for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the function $j_A(\tau) = c\tau + d$ defines a cocycle $j \in Z^1_U(\Gamma, \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}})$.
- For $\Gamma = \Gamma_{\mathbf{r}}$ and $U_A = \mathbb{C} \setminus \{\tau \in \mathbb{R} : j_A(\tau) \leq 0\}$, the function $ש^{\mathbf{r}}_A(\tau)$ defines a cocycle $ש^{\mathbf{r}} \in Z^1_U(\Gamma, \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}})$, which we call the Shintani–Faddeev modular cocycle.
- We will see that these define nontrivial cohomology classes $[j], [ש^{\mathbf{r}}] \in H^1_U(\Gamma, \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}})$.

Real multiplication (RM) values of a modular cocycle

- Let Γ be a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
- Let $w = (w_A)_{A\in\Gamma} \in H^1_U(\Gamma, \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}})$.
- Consider a real quadratic number β.
- Let $A \in \Gamma$ be the "positive" generator for the stabilizer of β in Γ (or in $\Gamma/\{\pm I\}$ if $-I \in \Gamma$), with $A \cdot \binom{\beta}{1} = \lambda \binom{\beta}{1}$ for $\lambda > 1$.
- If $\beta \in U_A$, call the value $w[\beta] := w_A(\beta)$ the real multiplication (RM) value of w at β.
- The value $w[\beta]$ depends only on β and the cohomology class $[w] \in H^1_U(\Gamma, \mathcal{M}^\times_{\mathbb{C},\mathbb{Q}})$.

RM values of the standard weight cocycle

Recall $j_A(\tau) = c\tau + d$ defines a modular cocycle for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Proposition (Exercise)

Let β be a real quadratic irrational and $\mathfrak{b} = \beta\mathbb{Z} + \mathbb{Z}$. Let

$$\mathcal{O} = (\mathfrak{b} : \mathfrak{b}) = \{\alpha \in \mathcal{O} : \alpha\mathfrak{b} \subseteq \mathfrak{b}\}$$

be the multiplier ring of $\mathfrak{b}$ (an order in $\mathbb{Q}(\beta)$). Then

$$j[\beta] = \varepsilon_{\mathcal{O}}^{+} \in \mathcal{O}^{\times}$$

where $\varepsilon_{\mathcal{O}}^{+} > 1$ generates the totally positive part of $\mathcal{O}^{\times}$.

Main theorem: RM values of the Shintani–Faddeev modular cocycle

Theorem (K 2024)

- Let F be an real quadratic number field.
- Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be a nonzero ideal with $\mathfrak{m} \neq \mathcal{O}_F$.
- Let $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m},\{\infty_2\}}$ and $\mathfrak{A}_0$ the class of $\mathfrak{A}$ in $\mathrm{Cl}_{\mathcal{O}_F,\emptyset}$.
- Choose $\mathfrak{b} \in \mathfrak{A}_0^{-1}$ coprime to $\mathfrak{m}$.
- Write $\mathfrak{b}\mathfrak{m} = \alpha(\beta\mathbb{Z} + \mathbb{Z})$, α totally positive, $\beta > \beta'$.
- Choose $\mathbf{r} = \left(\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right) \in \mathbb{Q}^2$ such that $(\alpha(r_2\beta - r_1))\mathfrak{b}^{-1} \in \mathfrak{A}$ and $r_2\beta' - r_1 > 0$.
- Let $n = 2$ if $\mathcal{O}_F$ has a positive unit that is $-1 \pmod{\mathfrak{m}}$, and let $n = 1$ otherwise.

Then, treating the characters ψ^2 and $\chi_{\mathbf{r}}$ as constant cocycles,

$$\exp\left(-nZ'_{\mathfrak{m},\{\infty_2\}}(0,\mathfrak{A})\right) = (\psi^2\,\chi_{\mathbf{r}}\,(\text{ш}^{\mathbf{r}})^{-2})[\beta].$$

RM values of the Shintani–Faddeev modular cocycle: proof outline

- Use Jacobi triple product and modularity of $\frac{\theta_{\mathbf{r}}(\tau)}{\eta(\tau)}$ to establish an elementary formula for $\mathfrak{w}_A^{\mathbf{r}}(\tau)\,\mathfrak{w}_A^{-\mathbf{r}}(\tau)$.
- Write Tangedal's version of Shintani's formula (involving the Hirzebruch–Jung continued fraction of β) in terms of $\mathfrak{w}_A^{\mathbf{r}}$, and use the cocycle condition to "telescope" the product.
- One is left with a complicated-looking root of unity factor...
- ...that may be simplified greatly using the combinatorics of continued fraction expansions and properties of metaplectic chracters.

Remark: Main theorem is a refinement not only of Shintani's limit formula but also of work of Arakawa (1982), Hayes (1990), Sczech (1995), Tangadal (2007), Yamamoto (2010).

RM values of the Shintani–Faddeev modular cocycle: example 1

Proposition (K 2024)

For any $A \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{C} \setminus \{\tau \in \mathbb{R} : j_A(\tau) \leq 0\}$,

$$\mathfrak{w}_A^{\binom{0}{1}}(\tau) = \mathrm{e}\left(\frac{-(A \cdot \tau) + \tau}{24}\right) \psi\left(A, \sqrt{j_A}\right) \sqrt{j_A(\tau)}.$$

Corollary (K 2024)

For β a real quadratic irrational, A be the positive generator for $\mathrm{stab}(\beta)$ in $\mathrm{SL}_2(\mathbb{Z})$, and $\mathcal{O} = (\beta\mathbb{Z} + \mathbb{Z} : \beta\mathbb{Z} + \mathbb{Z})$,

$$\mathfrak{w}^{\binom{0}{1}}[\beta] = \psi\left(A, \sqrt{j_A}\right) \sqrt{\varepsilon_{\mathcal{O}}^{+}}.$$

RM values of the Shintani–Faddeev modular cocycle: example 2

The following special value was computed to high precision and repeats an example from earlier in the talk. Equality is conditional on the Stark conjectures.

$$\mathfrak{W}^{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}\left[\sqrt{3}\right] = \mathfrak{W}^{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}_{\left(\begin{smallmatrix}26 & 45\\15 & 26\end{smallmatrix}\right)}\left(\sqrt{3}\right) \stackrel{?}{=} e^{-7\pi i/20}\sqrt{u}$$

where $u \approx 5.54061$ is a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

Some omissions from this presentation

- Full main theorem handles arbitrary RM values of $\mathfrak{w}^{\mathbf{r}}$, using partial zeta functions of ray class monoids of nonmaximal orders.
- Everything can rephrased in terms of RM values of the Shintani–Faddeev Jacobi cocycle $\sigma_{\mathbf{m},A}(z,\tau)$ for the Jacobi group $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.
- The first cohomology groups defined in this talk are the first cohomology of a certain Čech-like complex of "Γ-sheaves" after taking sheaf-theoretic Γ-invariants followed by global sections. Precise relationship to equivariant cohomology (if any) is TBD.
- Precise relationships to Eichler cohomology and quantum modularity are also TBD.

Thank you!

Thank you for listening! Any questions?