Analytic Number Theory: Problem Set 1

Due September 18, 2025, in class

Imagination is the Discovering Faculty, pre-eminently. It is that which penetrates into the unseen worlds around us, the worlds of Science. It is that which feels & discovers what is, the real which we see not, which exists not for our senses. Those who have learned to walk on the threshold of the unknown worlds, by means of what are commonly termed par excellence the exact sciences, may then with the fair white wings of Imagination hope to soar further into the unexplored amidst which we live. –Ada Lovelace

(1) [5 points] Prove the *summation by parts* formula, as follows. If $\langle a_k \rangle$ and $\langle b_k \rangle$ are sequences taking values in some commutative ring, then

$$\sum_{n=M}^{N} a_n(b_n - b_{n-1}) = a_N b_N - a_{M-1}b_{M-1} - \sum_{n=M}^{N} (a_n - a_{n-1})b_{n-1}.$$

(2) Define an *arithmetic function* to be a function $f : \mathbb{N} \to \mathbb{C}$. (Here $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. We use the term "arithmetic function" rather than "sequence" to specify that a function will be written with function notation, as well as to bring to mind certain associations with the term that will be developed in this problem.) If f and g are arithmetic functions, the *convolution* of f and g is defined by the following sum over positive divisors of n:

$$(f * g) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

If f is an arithmetic function, the *Dirichlet series associated to* f is defined to be

$$L(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

(a) [5 points] Prove that arithmetic functions form a commutative ring $\mathcal{A}$ with unity under addition and convolution. Denote by ι the multiplicative identity of this ring (and be sure to work out a formula for $\iota(n)$ in your proof).

(b) [3 points] Prove that $L(f * g, s) = L(f, s)L(g, s)$ for all s for which that Dirichlet series $L(f, s)$ and $L(g, s)$ converge absolutely.

(c) [3 points] Define the arithmetic functions

$$\mathbb{1}(n) = 1, \qquad \mu(n) = \begin{cases} 1 & \text{if } n \text{ is squarefree and has a even number of prime factors,} \\ -1 & \text{if } n \text{ is squarefree and has a odd number of prime factors,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Prove that $\mathbb{1}$ and μ are inverses in $\mathcal{A}$. (This statement is called Möbius inversion.)

(d) [3 points] Define the arithmetic functions

$$\mathrm{id}(n) = n, \qquad \varphi(n) = |\{a \in \mathbb{Z} : 0 \le a < n, \gcd(a, n) = 1\}|.$$

Prove that $\varphi * \mathbb{1} = \mathrm{id}$.

(e) [6 points] Define $\sigma_k(n) = \sum_{d|n} d^k$. Find and prove a formulas for $L(\iota, s)$, $L(\mathbb{1}, s)$, $L(\mu, s)$, $L(\mathrm{id}, s)$, $L(\varphi, s)$, and $L(\sigma_k, n)$ in terms of the Riemann zeta function.

(3) [5 points] Let $f(x)$ be a continuous function and $G(x) = \sum_{\substack{n\in\mathbb{N}\\ n\le x}} g(n)$ for some arithmetic function $g(n)$. For $0 < a \le b$, prove that the Reimann–Stieltjes integral

$$\int_a^b f(x)\,dG(x) = \sum_{\substack{n\in\mathbb{N}\\ a<n\le b}} f(n)g(n).$$

(4) [8 points] Prove *Shapiro's Tauberian Theorem*, as follows. Let $\langle a_n\rangle$ be an infinite sequence of nonnegative real numbers with the property that

$$B(x) := \sum_{\substack{n\in\mathbb{N}\\ n\le x}} a_n \left\lfloor \frac{x}{n} \right\rfloor = x\log x + Cx + o(x) \text{ as } x \to \infty,$$

where C is a real constant. Let $A(x) = \sum_{\substack{n\in\mathbb{N}\\ n\le x}} a_n$. Prove that there exist two positive constants α, β, and x_0 such that

$$\alpha x \le A(x) \le \beta x$$

for all $x > x_0$. Determine precise values for α and β.

Hints: Treat first the case that $C = 0$. Consider $A(x) - A\left(\frac{x}{2}\right)$ and use properties of the function $f(x) = \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor$.

(5) Define the *von Mangoldt function*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a prime power,} \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Prove that $\sum_{d|n} \Lambda(d) = \log n$.

(b) [4 points] Prove that, for $x \ge 2$,

$$\sum_{\substack{n\in\mathbb{N}\\ n\le x}} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor = x\log x - x + O(\log x).$$

(6) Using the results of preceding two problems, prove the following statements.

(a) [2 points] Let $C(x) = \sum_{\substack{n\in\mathbb{N}\\ n\le x}} \Lambda(n)$. Find explicit positive constants α, β such that

$$\alpha x \le C(x) \le \beta x$$

for all sufficiently large x.

(b) [3 points] Deduce from (a) a Chebyshev-type estimate by using summation by parts or Reimann–Stieltjes integration. That is, determine explicit positive constants a, b such that

$$a\,\frac{x}{\log x} \le \pi(x) \le b\,\frac{x}{\log x},$$

where $\pi(x)$ is the prime counting function.